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Hydrodynamics and transport coefficients for dilute granular gases

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The hydrodynamics of granular gases of viscoelastic particles, whose collision is described by an impact-velocity dependent coefficient of restitution, which is developed using the Chapman-Enskog approach. We derive the hydrodynamic equations and the according transport coefficients with the assumption that the shape of the velocity distribution function follows adiabatically the decaying temperature. We show numerically that this approximation is justified up to intermediate dissipation. The transport coefficients and the coefficient of cooling are expressed in terms of the elastic and dissipative parameters of the particle material and by the gas parameters. The dependence of these coefficients on temperature differs qualitatively from that obtained with the simplifying assumption of a constant coefficient of restitution which was used in previous studies. The approach formulated for gases of viscoelastic particles may be applied also for other impact-velocity dependencies of the restitution coefficient.

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I. INTRODUCTION

Granular systems composed of a large number of dissipatively interacting particles behave in many respects as a continuous medium and may be, in principle, described by a set of hydrodynamic equations with appropriate boundary conditions. Although this approach is successively used in various fields of engineering and soil mechanics (e.g., Refs. [1,2]) a first-principles theory for dense granular media is still lacking. Hydrodynamics may be also applied to much simpler systems, such as rarefied granular gases. It has been used to describe many different processes, e.g., rapid granular flows, structure formation, etc. (see Ref. [3] for an overview). For these systems, the hydrodynamic equations are not postulated but derived from the Boltzmann equation. The corresponding transport coefficients are not phenomenological constants, instead they are obtained by regular methods, such as the Grad method [4] or the Chapman-Enskog method [5]. In most of the studies, which address the derivation of hydrodynamic equations and kinetic coefficients, it was assumed that the coefficient of restitution ε is a material constant, e.g., Refs. [6–13].

This assumption simplifies the analysis enormously; however, it is neither in agreement with experimental observations (e.g., Refs. [14–16]) nor with basic mechanics of particle collisions [17]. The coefficient of restitution depends on the impact velocity g and tends to unity for very small g. As a consequence, particles behave more and more elastically as the average velocity of the grains decreases. The simplest collision model which accounts for dissipative material deformation, is the model of viscoelastic particles. It is assumed in this model that the elastic stress in the bulk of the particle material depends linearly on the strain, whereas the dissipative stress depends linearly on the strain rate.

Based on the fundamental work by Hertz [18], the interaction force between colliding viscoelastic spheres has been derived [19]. Using this generalized Hertz law the coefficient of restitution of viscoelastic particles can be given as a function of the impact velocity and material parameters [20]:

$$\varepsilon = 1 - C_1 A \alpha^{2/5} g^{1/5} + \tfrac{3}{5} C_1^2 A^2 \alpha^{4/5} g^{2/5} \mp \cdots, \tag{1}$$

with

$$g \equiv |\vec{e}\cdot(\vec{v}_1 - \vec{v}_2)| = |\vec{e}\cdot\vec{v}_{12}|. \tag{2}$$

The unit vector $\vec{e} = \vec{r}_{12}/r_{12}$ specifies the collision geometry, i.e., the relative position $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ of the particles at the collision instant. Their precollision velocities are given by $\vec{v}_1$ and $\vec{v}_2$. The elastic constant

$$\alpha = \left(\frac{3}{2}\right)^{3/2} \frac{Y\sqrt{R^{\mathrm{eff}}}}{m^{\mathrm{eff}}(1-\nu^2)} \tag{3}$$

depends on the effective mass and radius $m^{\mathrm{eff}} \equiv m_1 m_2/(m_1 + m_2)$, $R^{\mathrm{eff}} \equiv R_1 R_2/(R_1 + R_2)$ of the colliding spheres, on the Young modulus Y, and on the Poisson ratio ν of the particle material. The dissipative coefficient A is a function of dissipative and elastic constants (see Ref. [19] for details). Finally, C_1 is a numerical constant [17,20]

$$C_1 = \frac{\sqrt{\pi}}{2^{1/5} 5^{2/5}} \frac{\Gamma(3/5)}{\Gamma(21/10)} \approx 1.153\,44. \tag{4}$$

Equation (1) describes pure viscoelastic interaction. The assumption of viscoelastic deformation is justified if the impact velocity is not too large to avoid plastic deformation of the particles and not too small to neglect surface effects such as adhesion, van der Waals forces, etc. We also assume that the rotational degrees of freedom of particles may be neglected and consider a granular gas of identical particles in the absence of external forces. Then the coefficient of restitution gives the velocities of particles after a collision $\vec{v}_1^{\,\prime}$, $\vec{v}_2^{\,\prime}$ in terms of their values before the collision:

$$\vec{v}_{1/2}^{\,\prime} = \vec{v}_{1/2} \mp \frac{1+\varepsilon}{2} (\vec{e}\cdot\vec{v}_{12})\vec{e}. \tag{5}$$

The impact-velocity dependence of the coefficient of restitution implies serious consequences for the granular gasdynamics: For the simplified case $\varepsilon=\text{const}$, the form of the velocity distribution function $f(\vec{v},t)$ is characterized by a time-independent scaled function $\tilde{f}(\vec{c})$. It depends only on the scaled velocity $\vec{c}=\vec{v}/v_T$, where $v_T(t)=\sqrt{2T(t)/m}$ is the thermal velocity and $T(t)$ is the granular temperature [21–23]. The distribution function depends on time only via the time dependence of the granular temperature, its shape is time independent. The small deviations of the velocity distribution function from the Maxwell distribution are determined by the time-independent coefficient of restitution. For granular gases of viscoelastic particles, however, the effective value of ε changes with time along with the thermal velocity $v_T(t)$, which gives a typical velocity of the particles. Therefore, the shape of the velocity distribution function evolves in a rather complicated way [24].

The time dependence of the velocity distribution function has to be taken into account when the hydrodynamic equations and the transport coefficients are derived. Since the simple scaling is violated, the standard methods of kinetic theory of gases, developed for $\varepsilon=\text{const}$ (e.g., Refs. [10,12,25]) must be revised.

As it follows from our analysis, the transport coefficients for gases of viscoelastic particles depend on temperature and time rather differently as compared with the case $\varepsilon=\text{const}$. Correspondingly, the behavior of gases of viscoelastic particles differs qualitatively from that of gases of particles with the simplified collision model $\varepsilon=\text{const}$.

We wish to remark that the impact-velocity dependence of ε has been already taken into account in Ref. [26] for the hydrodynamic description of granular shear flow. In this study, an empirical expression of the coefficient of restitution was applied and a Maxwellian velocity distribution was assumed. Moreover, the authors have used the standard Chapman-Enskog method without the modifications required for gases of dissipatively colliding particles. These modifications for the case of $\varepsilon=\text{const.}$ have been extensively elaborated in Ref. [10].

The aim of the present study is to develop a continuum description of granular gases of viscoelastic particles. We derive the hydrodynamic equations along with the transport coefficients and the coefficient of cooling. In the rest of the paper, in Secs. II–VI, we discuss in detail the most important case of the three-dimensional gases, while in Sec. VII, we present the results for the two-dimensional systems, which are frequently addressed in molecular dynamics studies.

II. VELOCITY DISTRIBUTION AND TEMPERATURE IN THE HOMOGENEOUS COOLING STATE

A. Evolution equations for temperature and for the second Sonine coefficient

In this section, we introduce the notations and briefly sketch the necessary results for the homogeneous cooling state; more detail may be found in Refs. [24,27].

The Boltzmann equation for a granular gas of viscoelastic particles in the homogeneous cooling state reads [24,27]

$$\frac{\partial}{\partial t} f(\vec{v}_1,t)=\sigma^2\int d\vec{v}_2\int d\vec{e}\,\Theta(-\vec{v}_{12}\cdot\vec{e})|\vec{v}_{12}\cdot\vec{e}|\times[\chi f(\vec{v}_1'',t)f(\vec{v}_2'',t)-f(\vec{v}_1,t)f(\vec{v}_2,t)]\equiv I(f,f), \tag{6}$$

where $\sigma=2R$ is the particle diameter. The velocities $\vec{v}_1''$ and $\vec{v}_2''$ denote the precollision velocities of the inverse collision, which leads to the after-collision velocities $\vec{v}_1$ and $\vec{v}_2$. The factor $|\vec{v}_{12}\cdot\vec{e}|$ characterizes the length of the collision cylinder of cross section σ^2 and the Heaviside step function $\Theta(-\vec{v}_{12}\cdot\vec{e})$ assures that only approaching particles collide. Finally, the factor χ in the gain term accounts for the Jacobian of the transformation $(\vec{v}_1'',\vec{v}_2'')\rightarrow(\vec{v}_1,\vec{v}_2)$ and for the ratio of the lengths of the collision cylinders $|\vec{v}_{12}''\cdot\vec{e}|/|\vec{v}_{12}\cdot\vec{e}|$ for the direct and the inverse collision. For the case of spheres colliding with a constant coefficient of restitution $\chi=1/\varepsilon^2$, while for viscoelastic spheres, with $\varepsilon=\varepsilon(g)$ given by Eq. (1), it reads [24,27]

$$\chi=1+\tfrac{11}{5}C_1A\alpha^{2/5}|\vec{v}_{12}\cdot\vec{e}|^{1/5}+\tfrac{66}{25}C_1^2A^2\alpha^{4/5}|\vec{v}_{12}\cdot\vec{e}|^{2/5}+\cdots. \tag{7}$$

The dependence of χ on the impact velocity does not allow to derive from the Boltzmann equation a time-independent equation for the scaled distribution function $\tilde{f}$. Contrary to the case of $\varepsilon=\text{const}$ [21–23], the scaled distribution function depends explicitly on time. Therefore, we write for a gas of viscoelastic particles

$$f(\vec{v},t)=\frac{n}{v_T^3(t)}\tilde{f}(\vec{c},t), \quad \vec{c}=\frac{\vec{v}}{v_T(t)}, \tag{8}$$

with the number density of the granular gas n and the thermal velocity $v_T(t)$ defined by the granular temperature:

$$\frac{3}{2}nT(t)=\int d\vec{v}\frac{mv^2}{2}f(\vec{v},t)=\frac{3}{2}n\frac{mv_T^2(t)}{2}. \tag{9}$$

Hence, the shape of the velocity distribution function, characterized by the rescaled function $\tilde{f}(\vec{c},t)$, does not persist but evolves along with temperature [24,27]. We wish to stress that the time dependence of $\tilde{f}(\vec{c},t)$ is caused by the dependence of the factor χ on the impact velocity. Contrary, for a gas of simplified particles $(\varepsilon=\text{const})$, we obtain $\chi=1/\varepsilon^2=\text{const}$ and, therefore, the rescaled distribution function is time independent, $\tilde{f}(\vec{c},t)=\tilde{f}(\vec{c})$.

For slightly dissipative particles, the velocity distribution function is close to the Maxwell distribution. It may be described by a Sonine polynomial expansion [8,22,23,28]:

$$\tilde{f}(\vec{c},t)=\phi(c)\left(1+\sum_{p=1}^{\infty}a_p(t)S_p(c^2)\right), \tag{10}$$

where $\phi(c)\equiv\pi^{-3/2}\exp(-c^2)$ is the scaled Maxwell distribution, $S_p(x)$ are the Sonine polynomials

$$S_0(x)=1,$$
$$S_1(x)=-x^2+\tfrac{3}{2},$$
$$S_2(x)=\frac{x^2}{2}-\frac{5x}{2}+\frac{15}{8}, \qquad \text{etc.,} \tag{11}$$

and $a_k(t)$ are the *time-dependent* Sonine coefficients, which characterize the form of the velocity distribution [23,24]. The first Sonine coefficient is trivial, $a_1=0$, due to the definition of temperature [22,23], while the other coefficients quantify deviations of the moments $\langle c^k\rangle$ of the velocity distribution from the moments of the Maxwell distribution $\langle c^k\rangle_0$, e.g.,

$$a_2=\frac{\langle c^4\rangle-\langle c^4\rangle_0}{\langle c^4\rangle_0}. \tag{12}$$

Thus, the first nontrivial Sonine coefficient a_2 characterizes the fourth moment of the distribution function.

For small enough inelasticity ($\varepsilon\gtrsim 0.6$), the distribution function is well approximated by the second Sonine coefficient a_2 [22,23,28], i.e., higher coefficients $a_k=0$ for $k\geqslant 3$ may be neglected. With this approximation the evolution a granular gas of viscoelastic particles in the homogeneous cooling state is described by a set of coupled equations for the granular temperature and for the second Sonine coefficient [24,27]:

$$\frac{dT}{dt}=-\frac{2}{3}BT\mu_2\equiv-\zeta T, \tag{13}$$

$$\frac{da_2}{dt}=\frac{4}{3}B\mu_2(1+a_2)-\frac{4}{15}B\mu_4. \tag{14}$$

The coefficient $B\equiv B(t)=v_T(t)\sigma^2 g_2(\sigma)n$ is proportional to the mean collision frequency. The moments of the collision integral, μ_2 and μ_4 read with the approximation $\tilde{f}=\phi(c)[1+a_2(t)S_2(c^2)]$:

$$\mu_p=-\frac{1}{2}\int d\vec{c}_1\int d\vec{c}_2\int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|\phi(c_1)\phi(c_2)$$
$$\times\{1+a_2[S_2(c_1^2)+S_2(c_2^2)]+a_2^2S_2(c_1^2)S_2(c_2^2)\}$$
$$\times\Delta(c_1^p+c_2^p), \tag{15}$$

where

$$\Delta\psi(\vec{c}_i)\equiv\psi(\vec{c}_i{}')-\psi(\vec{c}_i) \tag{16}$$

denotes the change of some function $\psi(\vec{c}_i)$ according to a collision. The coefficients μ_p depend on time via the time dependence of a_2. For small enough dissipation, the moments μ_2 and μ_4 may be obtained as expansions in the time-dependent dissipative parameter δ',

$$\delta'(t)\equiv A\alpha^{2/5}[2T(t)]^{1/10}\equiv\delta\left[\frac{2T(t)}{T_0}\right]^{1/10}, \tag{17}$$

with $\delta\equiv A\alpha^{2/5}T_0^{1/10}$ and with the initial temperature T_0. These expansions read [24,27]

$$\mu_2=\sum_{k=0}^{2}\sum_{n=0}^{2}\mathcal{A}_{kn}\delta'^k a_2^n,$$
$$\mu_4=\sum_{k=0}^{2}\sum_{n=0}^{2}\mathcal{B}_{kn}\delta'^k a_2^n, \tag{18}$$

where $\mathcal{A}_{kn}$ and $\mathcal{B}_{kn}$ are numerical coefficients. They may be written in the compact matrix notation (rows refer to the first index)

$$\hat{\mathcal{A}}=\begin{pmatrix} 0 & 0 & 0 \\ \omega_0 & \frac{6}{25}\omega_0 & \frac{21}{2500}\omega_0 \\ -\omega_1 & -\frac{119}{400}\omega_1 & -\frac{4641}{640\,000}\omega_1 \end{pmatrix},$$

$$\hat{\mathcal{B}}=\begin{pmatrix} 0 & 4\sqrt{2\pi} & \frac{1}{8}\sqrt{2\pi} \\ \frac{28}{5}\omega_0 & \frac{903}{125}\omega_0 & -\frac{567}{12500}\omega_0 \\ -\frac{77}{10}\omega_1 & -\frac{476\,973}{44\,000}\omega_1 & \frac{4\,459\,833}{7\,040\,0000}\omega_1 \end{pmatrix} \tag{19}$$

with

$$\omega_0\equiv 2\sqrt{2\pi}2^{1/10}\Gamma\left(\frac{21}{10}\right)\approx 6.485,$$
$$\omega_1\equiv\sqrt{2\pi}2^{1/5}\Gamma\left(\frac{16}{5}\right)C_1^2\approx 9.285. \tag{20}$$

The coupled equations (13) and (14) together with Eqs. (17)–(19) determine the evolution of a granular gas of viscoelastic particles in the homogeneous cooling state. In particular, they define the velocity distribution function which is the starting point for the investigation of inhomogeneous gases.

In the limit of small dissipation, $\delta\ll 1$, the coupled equations (13) and (14) may be solved analytically. In linear approximation with respect to δ the solution reads [24,27]

$$\frac{T(t)}{T_0}=\left(1+\frac{t}{\tau_0}\right)^{-5/3}, \tag{21}$$

where we introduce the characteristic time

$$\tau_0^{-1}=\frac{16}{5}q_0\delta\tau_c(0)^{-1}=\frac{48}{5}q_0\delta 4\sigma^2 n\sqrt{\frac{\pi T_0}{m}}, \tag{22}$$

with the initial mean collision time $\tau_c(0)$ and the constant

$$q_0 = 2^{1/5}\Gamma\left(\frac{21}{10}\right)\frac{C_1}{8} \approx 0.173. \tag{23}$$

Correspondingly in linear approximation, the second Sonine coefficient depends on time as [24]

$$a_2(t) = -\tfrac{12}{5} w(t)^{-1}\{\mathrm{Li}(w(t)) - \mathrm{Li}(w(0))\}, \tag{24}$$

with

$$w(t) \equiv \exp\left[(q_0\delta)^{-1}\left(1+\frac{t}{\tau_0}\right)^{1/6}\right], \tag{25}$$

and with the logarithmic integral

$$\mathrm{Li}(x) \equiv \int_0^x \frac{1}{\ln(t)}\,dt. \tag{26}$$

For small dissipation δ, the time dependence of a_2 reveals two different regimes: (i) fast initial relaxation on the mean collision-time scale $\sim \tau_c(0)$ and (ii) subsequent slow evolution on the time scale $\sim \tau_0 \gg \tau_c(0)$, i.e., on the time scale of the temperature evolution. Therefore, the coefficient a_2 (and hence the form of the velocity distribution function) evolves in accordance with temperature. As we show below, the linear theory being rather accurate for small dissipation looses its accuracy with increasing δ (see Figs. 1 and 2). It may be, however, successively substituted by the adiabatic theory, addressed in the following subsection.

B. Adiabatic approximation for the second Sonine coefficient

For the hydrodynamic description of granular gases, we assume that there exist well separated time and length scales. The short time and length scales are given by the mean collision time and the mean free path, and the long time and length scales are characterized by the evolution of the hydrodynamic fields (to be defined in the following section) and their spatial inhomogeneities. The hydrodynamic approach corresponds to the coarse-grained description of the system, where all processes which take place on the short time and length scales are neglected. Therefore, the first stage of the relaxation of the velocity distribution function does not affect the hydrodynamic description and only the second stage of its evolution on the time scale $\tau_0 \gg \tau_c(0)$ is to be taken into account. To this end, we apply an adiabatic approximation: We omit the term da_2/dt in the left-hand side of Eq. (14) which describes the fast relaxation and assume that a_2 is determined by the current values of μ_2 and μ_4 due to the present temperature. Hence, in the adiabatic approximation, a_2 is determined by

$$5\mu_2(1+a_2) - \mu_4 = 0. \tag{27}$$

Using Eq. (18) for μ_2, μ_4, we find a_2 as an expansion in the small parameter δ':

$$a_2 = a_{21}\delta' + a_{22}\delta'^2 + \cdots,$$

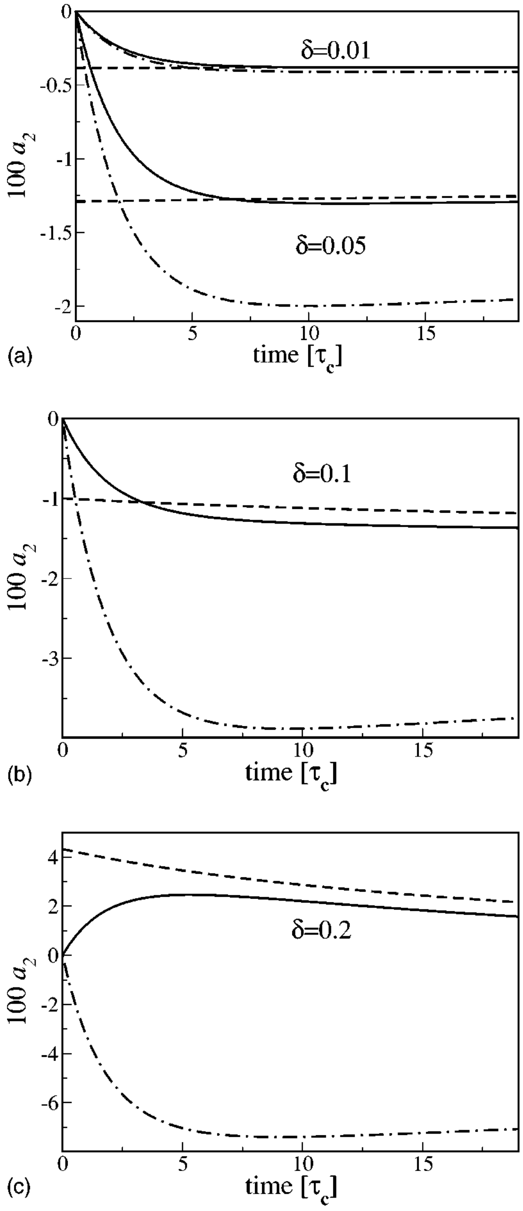

FIG. 1. Evolution of the second Sonine coefficient a_2 in the homogeneous cooling state on the short time scale. Solid line, numerical solution of Eqs. (13) and (14); dashed line, adiabatic approximation Eq. (28); dot-dashed line, the result of the linear theory, Eqs. (24) and (25). The time is given in collision units $\tau_c(0)$.

$$a_{21} = -\frac{3\omega_0}{20\sqrt{2\pi}} \approx -0.388,$$

$$a_{22} = \frac{12\,063}{640\,000}\frac{\omega_0^2}{\pi} + \frac{27}{40}\frac{\omega_1}{\sqrt{2\pi}} \approx 2.752. \tag{28}$$

Figures 1 and 2 show a_2 in adiabatic approximation due to Eq. (28) together with the numerical solution of Eqs. (13) and (14) and the results of the linear theory, Eqs. (24) and (25). The adiabatic approximation is rather accurate for small dissipation, $\delta < 0.05$, after the initial relaxation (5–10 colli-

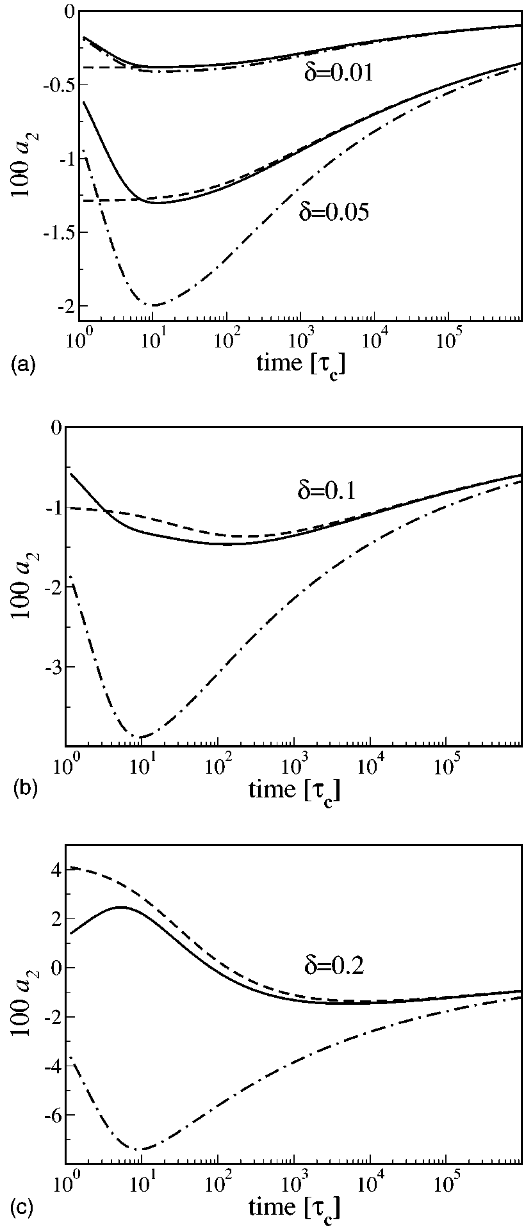

FIG. 2. Same as in Fig. 1, but for longer time. The adiabatic approximation improves as the system evolves.

sions per particle) has passed. Even for the larger dissipation ($\delta = 0.2$), it is in agreement with the numerical result. This value of the dissipative parameter corresponds to the initial coefficient of restitution $\varepsilon \approx 0.75$ for the thermal velocity. The adiabatic approximation becomes more and more accurate as the system evolves. Contrary, the linear theory fails with increasing δ and becomes qualitatively incorrect for larger δ. In Figs. 1 and 2, the evolution of the system has started from a Maxwell distribution, i.e., $a_2(0) = 0$. Figure 3 shows the relaxation of a_2 from certain initial conditions $a_2(0) \neq 0$ which correspond to non-Maxwellian distribution functions. The relaxation occurs during the first 5–10 collisions per particle for all considered values of δ. Then the adiabatic approximation becomes valid even for intermediate values of the dissipative parameter δ. The adiabatic approximation yields very accurate results for the evolution of temperature. It coincides almost perfectly with the numerical solution for the short and for the long time scales and for all values of the dissipative parameter $0 < \delta \leqslant 0.2$ (see Fig. 4). Thus, we conclude that the adiabatic approximation may be applied for the hydrodynamic description of granular gases. By means of Eq. (28) for the second Sonine coefficient, we can determine the moments μ_2, μ_4, and the cooling coefficient ζ,

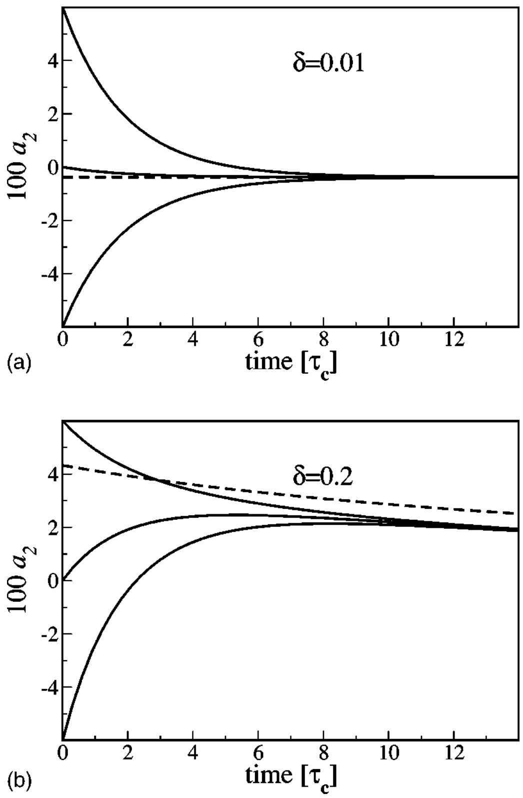

FIG. 3. Same as in Fig. 1, but for different initial values of a_2. The fast relaxation takes place during the first 5–10 collisions, independently on $a_2(0)$.

$$\zeta = \frac{2}{3}\mu_2 B = \frac{2}{3}\sqrt{\frac{2T}{m}}n\sigma^2(\omega_0\delta' - \omega_2\delta'^2 + \cdots), \tag{29}$$

where

$$\omega_2 \equiv \omega_1 + \frac{9}{500}\omega_0^2\sqrt{\frac{2}{\pi}}. \tag{30}$$

Below we will need the derivatives of a_2 and ζ with respect to temperature and density. Using

$$T\frac{\partial \delta'}{\partial T} = \frac{\delta'}{10}, \tag{31}$$

according to the definition of δ', Eq. (17), we obtain

$$T\frac{\partial a_2}{\partial T} = \frac{1}{10}a_{21}\delta' + \frac{1}{5}a_{22}\delta'^2, \tag{32}$$

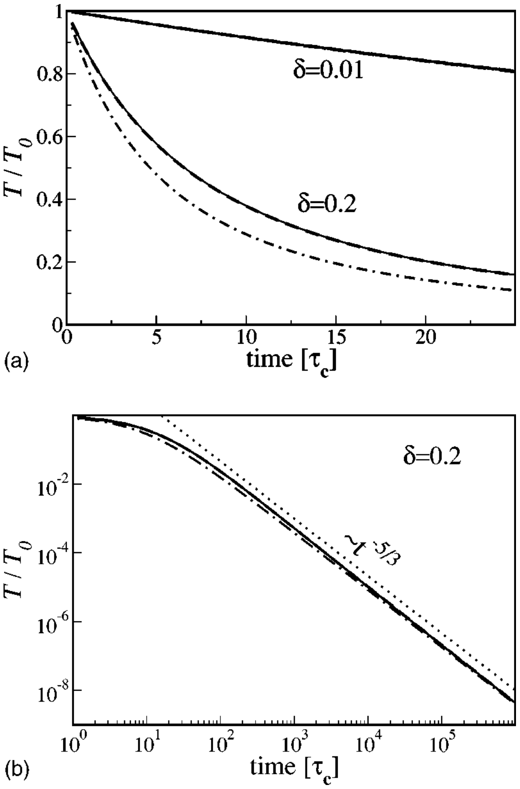

FIG. 4. Evolution of temperature in the homogeneous cooling state on the short time scale (top) and for longer time (bottom). The notations are the same as in Fig. 1. The dotted line shows the asymptotic power law at $t \to \infty$.

$$T\frac{\partial \zeta}{\partial T} = \frac{2}{3}B\left(\frac{3}{5}\omega_0 \delta' - \frac{7}{10}\omega_2 \delta'^2 + \cdots\right), \tag{33}$$

$$\frac{\partial \zeta}{\partial n} = \frac{1}{n}\zeta^{(0)}. \tag{34}$$

III. HYDRODYNAMIC EQUATIONS AND TRANSPORT COEFFICIENTS

A. Hydrodynamic fields and hydrodynamic equations

For the derivation of hydrodynamic equations, from the Boltzmann equation it is assumed that there exist well separated time and length scales. As already briefly discussed in the preceding section, in granular gases there are (at least) two sets of scales. The microscopic scales are characterized by the mean collision time and the mean free path. The macroscopic scales are given by the characteristic time of the evolution of the hydrodynamic fields and the size of their spatial inhomogeneities. Hence, scale separation means that the macroscopic fields vary very slowly in space and time if measured in the microscopic units. This assumption allows for a gradient expansion in space and time, i.e., the application of the Chapman-Enskog method [5].

The application of the Chapman-Enskog approach to granular gases is more sophisticated than its application to molecular gases: For molecular gases the unperturbed solution (the basic solution) for the velocity distribution function is the Maxwell distribution, while for granular gases the basic solution is the time-dependent velocity distribution of the homogeneous cooling state.

For gases of particles in the homogeneous cooling state, which collide with $\varepsilon = \mathrm{const}$, the velocity distribution function depends on time only through the time-dependent characteristic velocity. The shape of the velocity distribution function, given by $\tilde{f}$, is fixed by ε and does not depend on time. Hence, the evolution of temperature describes the evolution of the velocity distribution function exhaustively. For granular gases of viscoelastic particles the shape of the velocity distribution function, characterized by the Sonine coefficients a_k, is time dependent due to the time dependence of these coefficients. In the preceding section, we exemplified this property for the first nontrivial coefficient $a_2(t)$, assuming that for small dissipation higher coefficients may be disregarded. Therefore, the evolution of granular gases of viscoelastic particles is described by the time dependence of temperature, i.e., by the second moment of the velocity distribution function and by the time dependence of its higher-order even moments which are characterized by the Sonine coefficients a_2, a_3, etc. [see Eq. (12)]. Hence, the hydrodynamic description of granular gases whose particles collide with $\varepsilon = \varepsilon(g)$ requires an extended set of hydrodynamic fields which includes the higher-order moments. A regular approach to derive hydrodynamic equations for this extended set of fields is the Grad method [4]. Alternatively, for small dissipation an adiabatic approximation can be applied: It is assumed that the shape of the velocity distribution function, albeit varying in time, follows adiabatically the current temperature. Correspondingly, the higher-order moments are determined by the temperature too. With this approximation a closed set of hydrodynamic equations for density $n(\vec{r},t)$, velocity $\vec{u}(\vec{r},t)$, and temperature $T(\vec{r},t)$ may be derived. These fields are defined, respectively, by the zeroth, first, and second moments of the velocity distribution function:

$$n(\vec{r},t) = \int d\vec{v} f(\vec{r},\vec{v},t),$$

$$n(\vec{r},t)\vec{u}(\vec{r},t) = \int d\vec{v}\vec{v} f(\vec{r},\vec{v},t),$$

$$\frac{3}{2}n(\vec{r},t)T(\vec{r},t) = \int d\vec{v}\frac{1}{2}mV^2 f(\vec{r},\vec{v},t), \tag{35}$$

where $\vec{V} \equiv \vec{v} - \vec{u}(\vec{r},t)$. For small dissipation, the velocity distribution is well approximated by the second Sonine coefficients $a_2(t)$, i.e., higher-order coefficients are neglected. In adiabatic approximation, a_2 is determined by temperature according to Eq. (28).

Multiplying the Boltzmann equation for an inhomogeneous gas

$$\left(\frac{\partial}{\partial t} + \vec{v}_1 \cdot \vec{\nabla}\right) f(\vec{r},\vec{v}_1,t) = I(f,f), \tag{36}$$

correspondingly by v_1^0, $\vec{v}_1$, $mv_1^2/2$, and integrating over $d\vec{v}_1$, we obtain the hydrodynamic equations (see, e.g., Ref. [5])

$$\frac{\partial n}{\partial t}+\vec{\nabla}\cdot(n\vec{u})=0, \tag{37}$$

$$\frac{\partial \vec{u}}{\partial t}+\vec{u}\cdot\vec{\nabla}\vec{u}+(nm)^{-1}\vec{\nabla}\cdot\hat{P}=0, \tag{38}$$

$$\frac{\partial T}{\partial t}+\vec{u}\cdot\vec{\nabla}T+\frac{2}{3n}(\hat{P}:\vec{\nabla}\vec{u}+\vec{\nabla}\cdot\vec{q})+\zeta T=0. \tag{39}$$

The cooling coefficient ζ in the sink term ζT may be written as

$$\begin{aligned}\zeta(\vec{r},t)=&\frac{\sigma^2 m}{12nT}\int d\vec{v}_1\int d\vec{v}_2\int d\vec{e}\,\Theta\\&\times(-\vec{v}_{12}\cdot\vec{e})|\vec{v}_{12}\cdot\vec{e}|f(\vec{r},\vec{v}_1,t)\\&\times f(\vec{r},\vec{v}_2,t)(\vec{v}_{12}\cdot\vec{e})^2(1-\varepsilon^2).\end{aligned} \tag{40}$$

The pressure tensor $\hat{P}$ and the heat flux $\vec{q}$ are defined by

$$P_{ij}(\vec{r},t)=\int D_{ij}(\vec{V})f(\vec{r},\vec{v},t)d\vec{v}+p\,\delta_{ij},$$

$$\vec{q}(\vec{r},t)=\int \vec{S}(\vec{V})f(\vec{r},\vec{v},t)d\vec{v}, \tag{41}$$

where $p=nT$ is the hydrostatic pressure. The velocity tensor D_{ij} and the vector $\vec{S}$ read

$$D_{ij}(\vec{V})\equiv m\left(V_iV_j-\frac{1}{3}\delta_{ij}V^2\right), \tag{42}$$

$$\vec{S}(\vec{V})\equiv\left(\frac{mV^2}{2}-\frac{5}{2}T\right)\vec{V}. \tag{43}$$

The structure of the hydrodynamic equations, except for the cooling term ζT, coincides with those for molecular gases.

B. Chapman-Enskog approach

The system Eqs. (37)–(39) is closed by expressing the pressure tensor P_{ij} and the heat flux $\vec{q}$ in terms of the hydrodynamic fields and the fields gradients. To this end, we apply the Chapman-Enskog approach [5]. Application of this method to dilute granular gases with a constant ε has been elaborated in detail in Ref. [10]. Following the same lines as in Ref. [10] one can obtain relations for P_{ij} and $\vec{q}$, Eqs. (46), with the transport coefficients η, κ, μ, given by Eqs. (75)–(77) which are expressed in terms of functions $\vec{\alpha}$, $\vec{\beta}$, γ_{ij}; these functions are, in turn, the solutions of Eqs. (68)–(70). Since our notations differ from that of Ref. [10] and the corresponding equations for the functions $\vec{\alpha}$, $\vec{\beta}$, γ_{ij} are slightly different, we wish to sketch in the rest of this section the derivation of the above relations.

The Chapman-Enskog method is based on two important assumptions: The evolution of the distribution function is completely determined by the evolution of its first few moments, i.e., it depends on space and time only through the hydrodynamic fields:

$$f(\vec{r},\vec{v},t)=f[\vec{v},n(\vec{r},t),\vec{u}(\vec{r},t),T(\vec{r},t)]. \tag{44}$$

As the second precondition, it is assumed that the gas is only slightly inhomogeneous on the microscopic length scale which allows for a gradient expansion of the velocity distribution function

$$f=f^{(0)}+\lambda f^{(1)}+\lambda^2 f^{(2)}+\cdots, \tag{45}$$

where each power k of the formal parameter λ corresponds to the order k in the spatial gradient. Thus, $f^{(0)}$ refers to the homogeneous cooling state, $f^{(1)}$ corresponds to the linear approximation with respect to the fields gradients, $f^{(2)}$ is the solution with respect to quadratic terms in the field gradients, etc. With these assumptions the Boltzmann equation may be solved iteratively for each order in λ, together with the hydrodynamic equations for the moments of the velocity distribution function. The solution in zeroth order in λ yields the velocity distribution function for the homogeneous cooling state $f^{(0)}$ and the corresponding evolution of temperature. The function $f^{(0)}$ is then used to compute $\hat{P}$ and $\vec{q}$, yielding $\hat{P}^{(0)}=\delta_{ij}p$, $\vec{q}^{(0)}=0$, and the hydrodynamic equations for the ideal fluid. These first-order equations contain only linear gradient terms. Then $f^{(1)}$ may be found employing the first-order hydrodynamic equations and the distribution function $f^{(0)}$. The obtained $f^{(1)}$ as well as the corresponding expressions for $\hat{P}^{(1)}$ and $\vec{q}^{(1)}$ are linear in the field gradients:

$$P_{ij}=p\,\delta_{ij}-\eta\left(\nabla_i u_j+\nabla_j u_i-\frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{u}\right),$$

$$\vec{q}=-\kappa\vec{\nabla}T-\mu\vec{\nabla}n. \tag{46}$$

The transport coefficients η, κ, and μ in these equations are expressed in terms of $f^{(1)}$. Hence, within the Chapman-Enskog approach for each order in the gradient expansion a closed set of equations may be derived. Keeping only the first-order field gradients for the distribution function and, respectively, for the pressure tensor and for the heat flux as in Eq. (46), the Navier-Stokes hydrodynamics is obtained. Keeping next-order gradient terms corresponds to the Burnett or super-Burnett description. We will restrict ourselves to the Navier-Stokes level [30] and skip for simplicity of the notation the superscript "(1)" for $\hat{P}$ and $\vec{q}$ in the above equations.

The Chapman-Enskog scheme also assumes a hierarchy of time scales and, respectively, a hierarchy of time derivatives

$$\frac{\partial}{\partial t}=\frac{\partial^{(0)}}{\partial t}+\lambda\frac{\partial^{(1)}}{\partial t}+\lambda^2\frac{\partial^{(2)}}{\partial t}+\cdots, \tag{47}$$

where each order k in the time derivative, $\partial^{(k)}/\partial t \equiv \partial_t^{(k)}$, corresponds to the related order in the space gradient. Consequently, the higher the order in the space gradient, the slower is the according time variation. Using the formal expansion parameter λ, we can write the Boltzmann equation

$$\left(\frac{\partial^{(0)}}{\partial t} + \lambda \frac{\partial^{(1)}}{\partial t} + \cdots \lambda \vec{v}_1 \cdot \vec{\nabla} \right) (f^{(0)} + \lambda f^{(1)} + \cdots) = I[(f^{(0)} + \lambda f^{(1)} + \cdots),(f^{(0)} + \lambda f^{(1)} + \cdots)], \tag{48}$$

and collect terms of the same order in λ. The equation in zeroth order,

$$\frac{\partial^{(0)}}{\partial t} f^{(0)} = I(f^{(0)}, f^{(0)}), \tag{49}$$

coincides with Eq. (6) for the homogeneous cooling state. According to the Chapman-Enskog scheme, we obtain the velocity distribution function in zeroth order

$$f^{(0)}(\vec{v},\vec{r},t) = \frac{n(\vec{r},t)}{v_T^3(\vec{r},t)} [1 + a_2 S_2(c^2)] \phi(c), \tag{50}$$

where $\vec{c} = (\vec{v} - \vec{u})/v_T = \vec{V}/v_T$. The corresponding hydrodynamic equations in this order read

$$\frac{\partial^{(0)}}{\partial t} n = 0, \quad \frac{\partial^{(0)}}{\partial t} \vec{u} = 0, \quad \frac{\partial^{(0)}}{\partial t} T = -\zeta^{(0)} T, \tag{51}$$

where $\zeta^{(0)}$ is to be calculated using Eq. (40) with $f = f^{(0)}$. In this way, we reproduce the previous result (29).

Collecting terms $O(\lambda)$, we obtain

$$\frac{\partial^{(0)} f^{(1)}}{\partial t} + \left(\frac{\partial^{(1)}}{\partial t} + \vec{v}_1 \cdot \vec{\nabla} \right) f^{(0)} + J^{(1)}(f^{(0)}, f^{(1)}) = 0, \tag{52}$$

where we introduce

$$-J^{(k)}(f^{(0)}, f^{(k)}) \equiv I(f^{(0)}, f^{(k)}) + I(f^{(k)}, f^{(0)}). \tag{53}$$

The corresponding first-order hydrodynamic equations read

$$\begin{aligned} \frac{\partial^{(1)} n}{\partial t} &= -\vec{\nabla}(n\vec{u}), \\ \frac{\partial^{(1)} \vec{u}}{\partial t} &= -\vec{u} \cdot \vec{\nabla} \vec{u} - \frac{1}{nm} \vec{\nabla} p, \\ \frac{\partial^{(1)} T}{\partial t} &= -\vec{u} \cdot \vec{\nabla} T - \frac{2}{3} T \vec{\nabla} \cdot \vec{u} - \zeta^{(1)} T. \end{aligned} \tag{54}$$

The term $\zeta^{(1)}$ is found by substituting $f = f^{(0)} + \lambda f^{(1)}$ into Eq. (40) and collecting terms of the order $O(\lambda)$. These terms contain the factor $f^{(0)}(\vec{v},\vec{r},t) f^{(1)}(\vec{v},\vec{r},t)$ in the integrand. They vanish upon integration according to the different symmetry of the functions $f^{(0)}$ and $f^{(1)}$, thus, $\zeta^{(1)} = 0$ [31].

Since the distribution function $f^{(0)}$ is known, we can evaluate those terms in Eq. (52) which depend only on $f^{(0)}$:

$$\begin{aligned} \left(\frac{\partial^{(1)}}{\partial t} + \vec{v}_1 \cdot \vec{\nabla} \right) f^{(0)} = {} & \frac{\partial f^{(0)}}{\partial n} \left(\frac{\partial^{(1)} n}{\partial t} + \vec{v}_1 \cdot \vec{\nabla} n \right) \\ & + \frac{\partial f^{(0)}}{\partial \vec{u}} \cdot \left(\frac{\partial^{(1)} \vec{u}}{\partial t} + \vec{v}_1 \cdot \vec{\nabla} \vec{u} \right) \\ & + \frac{\partial f^{(0)}}{\partial T} \left(\frac{\partial^{(1)}}{\partial t} T + \vec{v}_1 \cdot \vec{\nabla} T \right). \end{aligned} \tag{55}$$

With $\vec{v}_1 = \vec{V} + \vec{u}$, the time derivatives of n, $\vec{u}$, and T given in Eq. (54), and the relations

$$\frac{\partial f^{(0)}}{\partial n} = \frac{1}{n} f^{(0)}, \quad \frac{\partial f^{(0)}}{\partial \vec{u}} = -\frac{\partial f^{(0)}}{\partial \vec{V}}, \tag{56}$$

which follow from Eq. (50), we recast Eq. (52) for $f^{(1)}$ into the form

$$\begin{aligned} \frac{\partial^{(0)} f^{(1)}}{\partial t} + J^{(1)}(f^{(0)}, f^{(1)}) = {} & f^{(0)}(\vec{\nabla} \cdot \vec{u} - \vec{V} \cdot \vec{\nabla} \ln n), \\ & + \frac{\partial f^{(0)}}{\partial T} \left(\frac{2}{3} T \vec{\nabla} \cdot \vec{u} - \vec{V} \cdot \vec{\nabla} T \right) \\ & + \frac{\partial f^{(0)}}{\partial V_i} \left((\vec{V} \cdot \vec{\nabla}) u_i - \frac{1}{nm} \nabla_i p \right), \end{aligned} \tag{57}$$

where we take into account $\zeta^{(1)} = 0$.

The right-hand side is known since it contains only $f^{(0)}$. It is convenient, however, to rewrite it in a form which shows explicitly the dependences on the fields gradients. Employing

$$\frac{1}{nm} \nabla_i p = \frac{T}{m} \nabla_i \ln T + \frac{T}{m} \nabla_i \ln n, \tag{58}$$

the right-hand side of Eq. (57) yields

$$\frac{\partial^{(0)} f^{(1)}}{\partial t} + J^{(1)}(f^{(0)}, f^{(1)}) = \vec{A} \cdot \vec{\nabla} \ln T + \vec{B} \cdot \vec{\nabla} \ln n + C_{ij} \nabla_j u_i, \tag{59}$$

with

$$\vec{A}(\vec{V}) = -\vec{V} T \frac{\partial f^{(0)}}{\partial T} - \frac{T}{m} \frac{\partial f^{(0)}}{\partial \vec{V}}, \tag{60}$$

$$\vec{B}(\vec{V}) = -\vec{V} f^{(0)} - \frac{T}{m} \frac{\partial}{\partial \vec{V}} f^{(0)}, \tag{61}$$

$$C_{ij}(\vec{V}) = \frac{\partial}{\partial V_i} (V_j f^{(0)}) + \frac{2T}{3} \delta_{ij} \frac{\partial f^{(0)}}{\partial T}. \tag{62}$$

We will calculate these terms below. From the form of the right-hand side of Eq. (59), we expect the form of its solution

$$f^{(1)}=\vec{\alpha}\cdot\vec{\nabla}\ln T+\vec{\beta}\cdot\vec{\nabla}\ln n+\gamma_{ij}\nabla_j u_i\,, \tag{63}$$

which is the most general form of a scalar function, which depends linearly on the vectorial gradients $\vec{\nabla}T$, $\vec{\nabla}n$, and on the tensorial gradients $\nabla_j u_i$. The coefficients $\vec{\alpha}$, $\vec{\beta}$, and γ_{ij} are functions of $\vec{V}$ and of the hydrodynamic fields n, $\vec{u}$, and T.

We derive now equations for the coefficients $\vec{\alpha}$, $\vec{\beta}$, and γ_{ij} by substituting $f^{(1)}$ as given by Eq. (63) into the first-order equation (59) and equating the coefficients of the corresponding gradients. To this end, we need $\partial_t^{(0)}f^{(1)}$ and, therefore, the time derivatives of the coefficients $\vec{\alpha}$, $\vec{\beta}$, and γ_{ij},

$$\frac{\partial^{(0)}\vec{\alpha}}{\partial t}=\frac{\partial\vec{\alpha}}{\partial T}\,\frac{\partial^{(0)}T}{\partial t}+\frac{\partial\vec{\alpha}}{\partial n}\,\frac{\partial^{(0)}n}{\partial t}+\frac{\partial\vec{\alpha}}{\partial u_i}\,\frac{\partial^{(0)}u_i}{\partial t}=-\zeta^{(0)}T\frac{\partial\vec{\alpha}}{\partial T}, \tag{64}$$

where we use Eqs. (51) in zeroth order. Similarly, we obtain

$$\frac{\partial^{(0)}\vec{\beta}}{\partial t}=-\zeta^{(0)}T\frac{\partial\vec{\beta}}{\partial T},\qquad \frac{\partial^{(0)}\gamma_{ij}}{\partial t}=-\zeta^{(0)}T\frac{\partial\gamma_{ij}}{\partial T}, \tag{65}$$

and, respectively, the time derivatives of the gradients

$$\frac{\partial^{(0)}}{\partial t}\vec{\nabla}\ln n=0,$$

$$\frac{\partial^{(0)}}{\partial t}\nabla_j u_i=0,$$

$$\frac{\partial^{(0)}}{\partial t}\vec{\nabla}\ln T=-\vec{\nabla}\zeta^{(0)}=-\left(\frac{\partial\zeta^{(0)}}{\partial n}\right)\vec{\nabla}n-\left(\frac{\partial\zeta^{(0)}}{\partial T}\right)\vec{\nabla}T. \tag{66}$$

The derivatives of $\zeta^{(0)}$ are given by Eqs. (33) and (34). From Eqs. (64)–(66), we obtain

$$\frac{\partial^{(0)}f^{(1)}}{\partial t}=-\left(T\frac{\partial\zeta^{(0)}}{\partial T}\vec{\alpha}\right)\cdot\vec{\nabla}\ln T-\left(\zeta^{(0)}T\frac{\partial\vec{\beta}}{\partial T}+\zeta^{(0)}\vec{\alpha}\right)\cdot\vec{\nabla}\ln n-\zeta^{(0)}T\frac{\partial\gamma_{ij}}{\partial T}\nabla_j u_i\,, \tag{67}$$

where we use Eq. (34). If we insert $\partial_t^{(0)}f^{(1)}$ into Eq. (59) and equate the coefficients of the gradients, we arrive at a set of equations for the coefficients $\vec{\alpha}$, $\vec{\beta}$, and γ_{ij},

$$-T\frac{\partial\zeta^{(0)}}{\partial T}\vec{\alpha}+J^{(1)}(f^{(0)},\vec{\alpha})=\vec{A}, \tag{68}$$

$$-\zeta^{(0)}T\frac{\partial\vec{\beta}}{\partial T}-\zeta^{(0)}\vec{\alpha}+J^{(1)}(f^{(0)},\vec{\beta})=\vec{B}, \tag{69}$$

$$-\zeta^{(0)}T\frac{\partial\gamma_{ij}}{\partial T}+J^{(1)}(f^{(0)},\gamma_{ij})=C_{ij}\,, \tag{70}$$

where $J^{(1)}$ is defined by Eq. (53).

C. Kinetic coefficients in terms of the velocity distribution function

From the definition of the pressure tensor Eq. (41) and its expression in terms of the field gradients, Eq. (46), follows:

$$\int D_{ij}(f^{(0)}+\vec{\alpha}\cdot\vec{\nabla}\ln T+\vec{\beta}\cdot\vec{\nabla}\ln n+\gamma_{kl}\nabla_l u_k)d\vec{V}=-\eta\left(\nabla_i u_j+\nabla_j u_i-\frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{u}\right), \tag{71}$$

where the tensor $D_{ij}(\vec{V})$ has been defined above. The integrals

$$\int D_{ij}f^{(0)}d\vec{V}=\int D_{ij}\vec{\alpha}d\vec{V}=\int D_{ij}\vec{\beta}d\vec{V}=0. \tag{72}$$

vanish since D_{ij} is a traceless tensor and $f^{(0)}$ depends isotropically on $\vec{V}$. Moreover, as will be shown below, the vectors $\vec{\alpha}$ and $\vec{\beta}$ are directed along $\vec{V}$, hence, the respective integrands are odd functions of $\vec{V}$. Therefore, only the term with the factor $\gamma_{kl}\nabla_l u_k$ on the left-hand side of Eq. (71) is nontrivial. Equating the coefficients of the gradient factor $\nabla_l u_k$, we obtain

$$\int D_{ij}\gamma_{kl}d\vec{V}=-\eta\left(\delta_{li}\delta_{kj}+\delta_{lj}\delta_{ki}-\frac{2}{3}\delta_{ij}\delta_{kl}\right). \tag{73}$$

For $k=j$, $l=i$, the last equation turns into

$$\int D_{ij}\gamma_{ji}d\vec{V}=-\eta\left(\delta_{ii}\delta_{jj}+\delta_{ij}\delta_{ij}-\frac{2}{3}\delta_{ij}\delta_{ij}\right)=-10\eta \tag{74}$$

($\delta_{ii}\delta_{jj}=9$ and $\delta_{ij}\delta_{ij}=3$ according to the summation convention) and yields the coefficients of viscosity

$$\eta=-\frac{1}{10}\int D_{ij}(\vec{V})\gamma_{ji}(\vec{V})d\vec{V}. \tag{75}$$

Using Eq. (41) for the heat flux and its corresponding expression in terms of the field gradients, Eq. (46), we can perform a completely analogous calculation and arrive at the kinetic coefficients κ and μ,

$$\kappa=-\frac{1}{3T}\int d\vec{V}\vec{S}(\vec{V})\cdot\vec{\alpha}(\vec{V}), \tag{76}$$

$$\mu=-\frac{1}{3n}\int d\vec{V}\vec{S}(\vec{V})\cdot\vec{\beta}(\vec{V}), \tag{77}$$

with $\vec{S}$ defined by Eq. (43). The coefficient of thermal conductivity κ has the standard interpretation, while the other coefficient μ does not have an analog for molecular gases.

IV. COEFFICIENT OF VISCOSITY

The viscosity coefficient is related to the coefficient γ_{ij} [see Eq. (75)] which, in turn, is the solution of Eq. 70 with the coefficient C_{ij} on the right-hand side. Let us first find an explicit expression for C_{ij}.

According to Eq. (50), the velocity distribution function depends on temperature through the thermal velocity v_T and additionally through the second Sonine coefficient a_2. Hence, the temperature derivative of $f^{(0)}$ reads

$$\frac{\partial f^{(0)}}{\partial T}=\frac{1}{2T}\frac{\partial}{\partial \vec{V}}\cdot \vec{V} f^{(0)}+f_M(V)S_2(c^2)\frac{\partial a_2}{\partial T}, \tag{78}$$

with $f_M(V)$ being the Maxwell distribution

$$f_M=\frac{n}{v_T^3}\phi(c) \quad \text{with} \quad \phi(c)=\frac{1}{\pi^{3/2}}\exp(-c^2). \tag{79}$$

Using the relation

$$\frac{\partial f^{(0)}}{\partial V_i}=\frac{V_i}{V}\frac{\partial f^{(0)}}{\partial V}, \tag{80}$$

and Eq. (78), the coefficient C_{ij} reads

$$C_{ij}(\vec{V}) =\left(V_iV_j-\frac{1}{3}\delta_{ij}V^2\right)\frac{1}{V}\frac{\partial f^{(0)}}{\partial V}+\frac{2}{3}\delta_{ij}S_2(c^2)f_M(V)T\frac{\partial a_2}{\partial T}. \tag{81}$$

With

$$\frac{1}{V}\frac{\partial f^{(0)}}{\partial V}=-\frac{m}{T}[1+a_2S_2(c^2)]f_M+\frac{m}{T}\left(c^2-\frac{5}{2}\right)a_2f_M, \tag{82}$$

we obtain

$$C_{ij}=-\frac{1}{T}D_{ij}\left[1+a_2\left(S_2(c^2)+\frac{5}{2}-c^2\right)\right]f_M(V) +\frac{2}{3}\delta_{ij}S_2(c^2)f_M(V)T\frac{\partial a_2}{\partial T}, \tag{83}$$

where $D_{ij}(\vec{V})$ has been defined in Eq. (42).

The expression for C_{ij} determines the right-hand side of Eq. (70) for γ_{ij} and hence, it suggests the form for γ_{ij}. For small dissipation (when a_2 is small) and for small fields gradients, we keep only the leading terms with respect to these variables. Therefore, we seek for γ_{ij} in the form

$$\gamma_{ij}(\vec{V})=\frac{\gamma_0}{T}D_{ij}(\vec{V})f_M(V), \tag{84}$$

where γ_0 is a velocity-independent coefficient, i.e., we neglect the dependence of γ_{ij} on a_2 [32]. The viscosity coefficient Eq. (75) reads then

$$\eta=-\frac{\gamma_0}{10}\frac{1}{T}\int d\vec{V}D_{ij}D_{ij}f_M=-\gamma_0 nT, \tag{85}$$

where we take into account that

$$D_{ij}D_{ji}=\tfrac{2}{3}m^2V^4 \tag{86}$$

according the definition of D_{ij}, Eq. (42), with the summation convention. Moreover, we have used the fourth moment of the Maxwell distribution. From Eq. (85) it follows:

$$\gamma_0=-\frac{\eta}{nT}. \tag{87}$$

Multiplying Eq. (70) by $D_{ij}(\vec{V}_1)$, integrating over $\vec{V}_1$ and using Eq. (75) yields

$$-10\zeta^{(0)}T\frac{\partial \eta}{\partial T}=-\int d\vec{V}_1D_{ij}(\vec{V}_1)C_{ij}(\vec{V}_1) +\int d\vec{V}_1D_{ij}(\vec{V}_1)J^{(1)}(f^{(0)},\gamma_{ij}). \tag{88}$$

To evaluate the first term on the right-hand side, we use Eq. (62) for C_{ij}, the relation

$$\frac{\partial}{\partial V_i}D_{ij}=\frac{\partial}{\partial V_i}m\left(V_iV_j-\frac{1}{3}\delta_{ij}V^2\right)=mV_j\left(1+\frac{1}{3}\delta_{ij}\right), \tag{89}$$

the definition of temperature, Eq. (35), and notice that $D_{ij}\delta_{ij}=0$ since D_{ij} is a traceless tensor [see Eq. (42)]. Integration by parts then yields

$$\begin{aligned}\int d\vec{V}_1D_{ij}C_{ij}&=\int d\vec{V}_1D_{ij}\frac{\partial}{\partial V_{1i}}V_{1j}f^{(0)} +\frac{2T}{3}\int d\vec{V}_1D_{ij}\delta_{ij}\frac{\partial f^{(0)}}{\partial T}\\ &=\int d\vec{V}_1f^{(0)}mV_{1j}V_{1j}\left(1+\frac{1}{3}\delta_{ij}\right)\\ &=\frac{10}{3}\int d\vec{V}_1f^{(0)}mV_1^2=10nT.\end{aligned} \tag{90}$$

For the second term on the right-hand side of Eq. (88), we use the definition of $J^{(1)}$, Eq. (53), and obtain

$$\int d\vec{V}_1D_{ij}J^{(1)}(f^{(0)},\gamma_{ij}) =-\int d\vec{V}_1D_{ij}I(f^{(0)},\gamma_{ij})-\int d\vec{V}_1D_{ij}I(\gamma_{ij},f^{(0)}). \tag{91}$$

We apply the property of the collision integral [5]

$$\int d\vec{V}_1 D_{ij} I(f^{(0)},\gamma_{ij}) = \int d\vec{V}_1 D_{ij} I(\gamma_{ij},f^{(0)}) = \frac{\sigma^2}{2}\int d\vec{V}_1 \int d\vec{V}_2 f^{(0)}(\vec{V}_1)\gamma_{ij}(\vec{V}_2) \times \int d\vec{e}\,\Theta(-\vec{V}_{12}\cdot\vec{e})|\vec{V}_{12}\cdot\vec{e}|\Delta \times [D_{ij}(\vec{V}_1)+D_{ij}(\vec{V}_2)], \tag{92}$$

where $\Delta\psi(\vec{v}_i)\equiv\psi(\vec{v}_i^{\,\prime})-\psi(\vec{v}_i)$ denotes as previously the change of some quantity $\psi(\vec{v}_i)$ due to a collision. Equation (91) turns then into

$$\int d\vec{V}_1 D_{ij} J^{(1)}(f^{(0)},\gamma_{ij}) = -\sigma^2\int d\vec{V}_1 \int d\vec{V}_2 f^{(0)}(\vec{V}_1)\gamma_{ij}(\vec{V}_2) \times \int d\vec{e}\,\Theta(-\vec{V}_{12}\cdot\vec{e})|\vec{V}_{12}\cdot\vec{e}|\Delta[D_{ij}(\vec{V}_1)+D_{ij}(\vec{V}_2)]. \tag{93}$$

We write the factors in the last integral using the dimensionless velocities $\vec{V}_{1/2}=v_T\vec{c}_{1/2}$:

$$D_{ij}(\vec{V}) = mv_T^2 D_{ij}(\vec{c}) = mv_T^2\left(c_ic_j-\frac{1}{3}\delta_{ij}c^2\right),$$

$$\gamma_{ij}(\vec{V}) = \frac{\gamma_0}{T}\left(\frac{n}{v_T^3}\right)mv_T^2 D_{ij}(\vec{c})\,\phi(c), \tag{94}$$

and recast Eq. (93) into the form

$$\int d\vec{V}_1 D_{ij}(\vec{V}_1) J^{(1)}(f^{(0)},\gamma_{ij}) = 4\eta v_T n\sigma^2\Omega_\eta, \tag{95}$$

where we substitute $\gamma_0=-\eta/nT$ and where Ω_η is a numerical coefficient defined by

$$\Omega_\eta \equiv \int d\vec{c}_1 \int d\vec{c}_2 \int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|\tilde{f}^{(0)}(c_1) \times \phi(c_2)D_{ij}(\vec{c}_2)\Delta[D_{ij}(\vec{c}_1)+D_{ij}(\vec{c}_2)]. \tag{96}$$

The coefficient Ω_η may be expressed in terms of the dissipation parameter δ' and the second Sonine coefficient a_2 (see the Appendix):

$$\Omega_\eta = -(w_0+\delta' w_1-\delta'^2 w_2), \tag{97}$$

with

$$w_0 = 4\sqrt{2\pi}\left(1-\frac{1}{32}a_2\right),$$

$$w_1 = \omega_0\left(\frac{1}{15}-\frac{1}{500}a_2\right),$$

$$w_2 = \omega_1\left(\frac{97}{165}-\frac{679}{44\,000}a_2\right). \tag{98}$$

Substituting Eqs. (90), (95), and (97) into Eq. (88) and using Eq. (29) for $\zeta^{(0)}$, we obtain an equation for the coefficient of viscosity η:

$$(\omega_0\delta'-\omega_2\delta'^2)T\frac{\partial\eta}{\partial T} = \frac{3}{5}(w_0+w_1\delta'-w_2\delta'^2)\,\eta-\frac{3}{2}\frac{1}{\sigma^2}\sqrt{\frac{mT}{2}}. \tag{99}$$

We seek the solution as an expansion in terms of δ':

$$\eta = \eta_0(1+\delta'\tilde{\eta}_1+\delta'^2\tilde{\eta}_2+\cdots). \tag{100}$$

The solution in zeroth order

$$\eta_0 = \frac{5}{16\sigma^2}\sqrt{\frac{mT}{\pi}} \tag{101}$$

is the viscosity coefficient for a gas of elastic particles (Enskog viscosity), while the coefficients $\tilde{\eta}_1$ and $\tilde{\eta}_2$ account for the dissipative properties of viscoelastic particles. With Eq. (31), the temperature derivative of the viscosity coefficient reads

$$T\frac{\partial\eta}{\partial T} = \eta_0\left(\frac{1}{2}+\frac{3}{5}\delta'\tilde{\eta}_1+\frac{7}{10}\delta'^2\tilde{\eta}_2+\cdots\right). \tag{102}$$

We substitute Eqs. (100) and (102) into Eq. (99), express a_2 in terms of δ' according to Eq. (28) and collect terms of the same order in δ'. This yields the equations for $\tilde{\eta}_1$, $\tilde{\eta}_2$, etc., whose solutions read

$$\tilde{\eta}_1 = \frac{359}{3840}\frac{\sqrt{2\pi}}{\pi}\omega_0 \approx 0.483,$$

$$\tilde{\eta}_2 = \frac{41\,881}{2304\,000}\frac{\omega_0^2}{\pi}-\frac{567}{28\,160}\frac{\omega_1\sqrt{2\pi}}{\pi} \approx 0.094, \tag{103}$$

with $\omega_{0/1}$ given by Eq. (20).

Thus, we arrive at the final expression for the viscosity coefficient for a granular gas of viscoelastic particles,

$$\eta = \frac{5}{16\sigma^2}\sqrt{\frac{mT}{\pi}}(1+0.483\delta'+0.094\delta'^2+\cdots). \tag{104}$$

In contrast to granular gases of simplified particles ($\varepsilon=\mathrm{const}$), where $\eta\propto\sqrt{T}$, for a gas of viscoelastic particles there is an additional temperature dependence due to the time-dependent coefficient δ'.

V. COEFFICIENT OF THERMAL CONDUCTIVITY AND THE COEFFICIENT μ

To find coefficients κ and μ, we need the coefficients $\vec{\alpha}$ and $\vec{\beta}$ which are the solutions of Eqs. (68) and (69). The functions $\vec{A}$ and $\vec{B}$ on the right-hand sides may be found from Eqs. (60) and (61):

$$\vec{A} = -\frac{1}{T}\vec{S}(\vec{V})[1 + a_2\{S_2(c^2) + 1 - c^2\}]f_M - \vec{V}S_2(c^2)\left[\frac{a_{21}}{10}\delta' + \frac{a_{22}}{5}\delta'^2\right]f_M, \tag{105}$$

$$\vec{B} = \frac{a_2}{T}\vec{S}(\vec{V})f_M. \tag{106}$$

Keeping only leading terms with respect to the gradients and a_2, we choose $\vec{\alpha}$ in the form

$$\vec{\alpha} = -\frac{\alpha_1}{T}\vec{S}(\vec{V})f_M(V), \tag{107}$$

with α_1 being the velocity independent coefficient. This ansatz for $\vec{\alpha}$ yields the coefficient of thermal conductivity

$$\kappa = -\frac{1}{3T}\int d\vec{V}\vec{S}(\vec{V})\cdot\vec{\alpha}(\vec{V}) = \frac{\alpha_1}{3T^2}\int d\vec{V}\left(\frac{mV^2}{2} - \frac{5}{2}T\right)^2 V^2 f_M = \frac{5}{2}\frac{nT}{m}\alpha_1, \tag{108}$$

which implies

$$\alpha_1 = \frac{2m}{5nT}\kappa. \tag{109}$$

Multiplying Eq. (68) for $\vec{\alpha}$ by $\vec{S}(\vec{V}_1)/T$, integrating over $\vec{V}_1$ and using Eq. (76) for κ, we obtain

$$3\frac{\partial}{\partial T}\zeta^{(0)}\kappa T = \frac{1}{T}\int d\vec{V}_1\vec{S}\cdot\vec{A} - \frac{1}{T}\int d\vec{V}_1\vec{S}\cdot J^{(1)}(f^{(0)},\vec{\alpha}). \tag{110}$$

To evaluate the first term on the right-hand side, we use Eq. (43) for $\vec{S}(\vec{V})$ and Eq. (105) for $\vec{A}(\vec{V})$:

$$-\frac{1}{T}\int d\vec{V}\vec{S}(\vec{V})\cdot\vec{A}(\vec{V}) = v_T^2\int d\vec{V}\left(c^2 - \frac{5}{2}\right)^2 c^2[1 + a_2\{S_2(c^2) + 1 - c^2\}]f_M + v_T^2\int d\vec{V}\left(c^2 - \frac{5}{2}\right)c^2\left[\frac{a_{21}}{10}\delta' + \frac{a_{22}}{5}\delta'^2\right]S_2(c^2)f_M. \tag{111}$$

With the Maxwell distribution equation (79), the first term in Eq. (111) reads

$$4\pi n v_T^2\left\{\int_0^\infty c^4\phi(c)\left(c^2 - \frac{5}{2}\right)^2 dc + a_2\int_0^\infty c^4\phi(c) \times\left(c^2 - \frac{5}{2}\right)^2\left[\frac{c^4}{2} - \frac{7c^2}{2} + \frac{23}{8}\right]dc\right\} = \frac{15}{4}nv_T^2 + \frac{15}{2}nv_T^2 a_2 = \frac{15}{2}\frac{nT}{m}(1 + 2a_2), \tag{112}$$

where integration over the angles has been performed and the integral

$$\int_0^\infty \exp(-x^2)x^{2k}dx = \frac{(2k-1)!!}{2^{k+1}}\sqrt{\pi} \tag{113}$$

was used. Very similar calculations give the second term in Eq. (111):

$$\frac{15}{4}nv_T^2\left(\frac{a_{21}}{10}\delta' + \frac{a_{22}}{5}\delta'^2\right). \tag{114}$$

Summing up Eqs. (112) and (114), we obtain the first term on the right-hand side of Eq. (110):

$$\frac{1}{T}\int d\vec{V}_1\vec{S}\cdot\vec{A} = -\frac{15}{2}\frac{nT}{m}\left(1 + \frac{21}{10}a_{21}\delta' + \frac{11}{5}a_{22}\delta'^2\right). \tag{115}$$

The second term on the right-hand side of Eq. (110) may be again written using the basic property of the collision integral [see Eq. (92)]:

$$-\frac{1}{T}\int d\vec{V}_1\vec{S}(\vec{V}_1)\cdot J^{(1)}(f^{(0)},\vec{\alpha}) = \frac{\sigma^2}{T}\int d\vec{V}_1\int d\vec{V}_2 f^{(0)}(\vec{V}_1)\vec{\alpha}(\vec{V}_2)\cdot\int d\vec{e} \times\Theta(-\vec{V}_{12}\cdot\vec{e})|\vec{V}_{12}\cdot\vec{e}|\Delta[\vec{S}(\vec{V}_1) + \vec{S}(\vec{V}_2)]. \tag{116}$$

Using the dimensionless variables

$$\vec{S}(\vec{V}) = v_T T\vec{S}(\vec{c}) = v_T T\left(c^2 - \frac{5}{2}\right)\vec{c},$$

$$\vec{\alpha}(\vec{V}) = -\alpha_1 v_T\left(\frac{n}{v_T^3}\right)\phi(c)\vec{S}(\vec{c}), \tag{117}$$

and Eq. (109) for α_1, we recast the last equation into the form

$$-\frac{1}{T}\int d\vec{V}\vec{S}(\vec{V})\cdot J^{(1)}(f^{(0)},\vec{\alpha}) = -\frac{4}{5}\kappa v_T n\sigma^2\Omega_\kappa. \tag{118}$$

The coefficient Ω_κ is defined by

$$\Omega_\kappa \equiv \int d\vec{c}_1 \int d\vec{c}_2 \int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|\tilde{f}^{(0)}(c_1)\times\phi(c_2)\vec{S}(\vec{c}_2)\cdot\Delta[\vec{S}(\vec{c}_1)+\vec{S}(\vec{c}_2)]. \tag{119}$$

This coefficient reads (see the Appendix)

$$\Omega_\kappa = -(u_0+\delta' u_1-\delta'^2 u_2), \tag{120}$$

where

$$u_0 = 4\sqrt{2\pi}\left(1+\frac{1}{32}a_2\right),$$
$$u_1 = \omega_0\left(\frac{17}{5}-\frac{9}{500}a_2\right),$$
$$u_2 = \omega_1\left(\frac{1817}{440}-\frac{1113}{352\,000}a_2\right). \tag{121}$$

Substituting Eqs. (115), (118), and (120) into Eq. (110) and using Eq. (29) for $\zeta^{(0)}$, we arrive at

$$T\frac{\partial}{\partial T}\kappa T^{3/2}(\omega_0\delta'-\omega_2\delta'^2+\cdots) = \frac{2}{5}\kappa T^{3/2}(u_0+\delta' u_1-\delta'^2 u_2+\cdots) - \frac{15}{4}\frac{T^{3/2}}{\sigma^2}\sqrt{\frac{T}{2m}}\left(1+\frac{21}{10}a_{21}\delta'+\frac{11}{5}a_{22}\delta'^2\right). \tag{122}$$

We solve this equation with the ansatz

$$\kappa = \kappa_0(1+\delta'\tilde{\kappa}_1+\delta'^2\tilde{\kappa}_2+\cdots), \tag{123}$$

where

$$\kappa_0 = \frac{75}{64\sigma^2}\sqrt{\frac{T}{\pi m}} \tag{124}$$

is the Enskog thermal conductivity for a gas of elastic particles. Substituting Eq. (123) into Eq. (122) and equating terms of the same order in δ', we obtain the coefficients

$$\tilde{\kappa}_1 = \frac{487}{6400}\frac{\sqrt{2\pi}\omega_0}{\pi} \approx 0.393,$$
$$\tilde{\kappa}_2 = \frac{1}{\pi}\left(\frac{2\,872\,113}{51\,200\,000}\omega_0^2+\frac{78\,939}{140\,800}\sqrt{2\pi}\omega_1\right) \approx 4.904. \tag{125}$$

Hence, the coefficient of thermal conductivity for a granular gas of viscoelastic particles reads in adiabatic approximation

$$\kappa = \frac{75}{64\sigma^2}\sqrt{\frac{T}{\pi m}}(1+0.393\delta'+4.904\delta'^2+\cdots). \tag{126}$$

Similar to the viscosity coefficient, the coefficient of thermal conductivity of a granular gas of viscoelastic particles reveals an additional temperature dependence as compared with gases of simplified particles where $\varepsilon = \text{const}$.

The evaluation of the coefficient μ may be performed in the same way as κ, i.e., choosing $\vec{\beta}$ in the form

$$\vec{\beta} = -\frac{\beta_1}{T}\vec{S}(\vec{V})f_M(V), \tag{127}$$

with the velocity independent coefficient β_1. The only difference is that the expansion of μ in terms of the dissipative parameter δ' lacks the term in zeroth order since μ vanishes in the elastic limit. Since the calculations are completely analogous to that for κ, we present here only the final result

$$\mu = \frac{\kappa_0 T}{n}(\delta'\tilde{\mu}_1+\delta'^2\tilde{\mu}_2+\cdots), \tag{128}$$

with

$$\tilde{\mu}_1 = \frac{19}{80}\frac{\omega_0\sqrt{2\pi}}{\pi} \approx 1.229,$$
$$\tilde{\mu}_2 = \frac{1}{\pi}\left(\frac{58\,813}{640\,000}\omega_0^2+\frac{1}{40}\sqrt{2\pi}\omega_1\right) \approx 1.415. \tag{129}$$

Thus, the coefficient μ reads in adiabatic approximation

$$\mu = \frac{\kappa_0 T}{n}(1.229\delta'+1.415\delta'^2+\cdots). \tag{130}$$

Finally, using the coefficients

$$\alpha_1 = \frac{2m}{5nT}\kappa, \quad \beta_1 = \frac{2m}{5T^2}\mu, \quad \gamma_0 = -\frac{1}{nT}\eta \tag{131}$$

(the result for β_1 may be derived analogously as for α_1) in the relations Eqs. (107), (127), and (84) for $\vec{\alpha}$, $\vec{\beta}$, γ_{ij}, we obtain an expression for the first-order distribution function $f^{(1)}$, which depends linearly on the field gradients according to Eq. (63).

VI. COOLING RATE

As it was already mentioned the cooling rate ζ in the zero-order approximation with respect to the fields gradients $\zeta^{(0)}$ coincides with that for the homogeneous cooling state, Eq. (29), while the first-order coefficient $\zeta^{(1)}$ is zero due to the different symmetry of the functions $f^{(0)}(\vec{V})$ and $f^{(1)}(\vec{V})$. For consistency of the hydrodynamic equations, which contain the second-order terms with respect to the fields gradients, it is necessary to include the second-order terms into the cooling rate ζ [10]. These coefficients appear in the second-order cooling coefficient

$$\zeta^{(2)} = \frac{\sigma^2 m}{12nT}\int d\vec{v}_1\int d\vec{v}_2\int d\vec{e}\,\Theta(-\vec{v}_{12}\cdot\vec{e})|\vec{v}_{12}\cdot\vec{e}|\times[f^{(1)}(\vec{v}_1)f^{(1)}(\vec{v}_2)+2f^{(0)}(\vec{v}_1)f^{(2)}(\vec{v}_2)]\times(\vec{v}_{12}\cdot\vec{e})^2(1-\varepsilon^2), \tag{132}$$

where the dependence on $\vec{r}$ and on t is suppressed for brevity. From the symmetry requirements, $\zeta^{(2)}$ depends on the fields gradients as follows [10]:

$$\zeta^{(2)}=\zeta_1\nabla^2 T+\zeta_2\nabla^2 n+\zeta_3(\vec{\nabla}T)^2+\zeta_4(\vec{\nabla}T)^2 +\zeta_5(\vec{\nabla}T)\cdot(\vec{\nabla}n)+\zeta_6\left[(\nabla_i u_j)(\nabla_i u_j)+(\nabla_i u_j)(\nabla_j u_i) -\frac{2}{d}(\vec{\nabla}\cdot\vec{u})^2\right]+\zeta_7(\vec{\nabla}\cdot\vec{u})^2, \tag{133}$$

where $d=3$ is the system dimension and the coefficients ζ_k $(k=1,2,\dots,7)$ may be calculated, provided the functions $f^{(0)}$, $f^{(1)}$,and $f^{(2)}$ are known. While $f^{(0)}$ and $f^{(1)}$ are given, respectively, by Eq. (50) and Eq. (63) with Eqs. (84), (107), (127), and (131), the function $f^{(2)}$ is still to be found. To derive $f^{(2)}$ one needs to solve the second-order equation of the Chapman-Enskog perturbation scheme:

$$\frac{\partial^{(0)}f^{(2)}}{\partial t}+J^{(2)}(f^{(0)},f^{(2)})+\frac{\partial^{(2)}f^{(0)}}{\partial t} =I(f^{(1)},f^{(1)})-\left[\frac{\partial^{(1)}}{\partial t}+\vec{u}\cdot\vec{\nabla}+\vec{V}\cdot\vec{\nabla}\right]f^{(1)}. \tag{134}$$

Here, we consider only the part of the function $f^{(2)}$ which contributes to $\zeta^{(2)}$. This part, which we denote $f_\zeta^{(2)}$ does not contain vectorial or traceless tensorial functions of the velocity, since this part of the function will vanish after integration over $\vec{v}_1$, $\vec{v}_2$, $\vec{e}$ in Eq. (132) [10]. Therefore, $f_\zeta^{(2)}$ has the form

$$f_\zeta^{(2)}=B_1\nabla^2 T+B_2\nabla^2 n+B_3(\vec{\nabla}T)^2+B_4(\vec{\nabla}T)^2 +B_5(\vec{\nabla}T)\cdot(\vec{\nabla}n)+B_6\left[(\nabla_i u_j)(\nabla_i u_j) +(\nabla_i u_j)(\nabla_j u_i)-\frac{2}{3}(\vec{\nabla}\cdot\vec{u})^2\right]+B_7(\vec{\nabla}\cdot\vec{u})^2, \tag{135}$$

where the coefficients B_k are scalar functions of n, T, and V^2. Moreover, they must be orthogonal to 1, $\vec{V}$, and V^2, according to the Chapman-Enskog scheme [5]. Therefore, their lowest-order Sonine expansion reads [10]

$$B_k(n,T,V)=b_k(n,T)S_2(c^2)f_M(V), \tag{136}$$

where, as previously, $c=V/v_T$. With the substitute Eq. (136), we can express the coefficients ζ_k in terms of b_k. For example, the first two coefficients ζ_1 and ζ_2 read

$$\zeta_1=B\Omega_\zeta^{(1)}b_1, \quad \zeta_2=B\Omega_\zeta^{(1)}b_2, \tag{137}$$

where B has been defined in Eq. (13) and $\Omega_\zeta^{(1)}$ is the numerical coefficient

$$\Omega_\zeta^{(1)}\equiv\int d\vec{c}_1\int d\vec{c}_2\int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}| \times\tilde{f}^{(0)}(c_1)\phi(c_2)(\vec{c}_{12}\cdot\vec{e})^2 S_2(c_2^2)(1-\varepsilon^2). \tag{138}$$

This coefficient reads (see the Appendix)

$$\Omega_\zeta^{(1)}=\left(\frac{12}{25}+\frac{21}{625}a_2\right)\omega_0\delta'-\left(\frac{119}{200}+\frac{4641}{160\,000}a_2\right)\omega_1\delta'^2. \tag{139}$$

The coefficients ζ_1 and ζ_2, which correspond to the linear part of $\zeta^{(2)}$ with respect to fields gradients have been derived in Ref. [10] for the case of a constant restitution coefficient. The authors conclude that these coefficients may be neglected for $\varepsilon=\text{const}$, if the inelasticity is not large. Following the approach in Ref. [10], we briefly sketch the derivation of these coefficients for the case of viscoelastic particles; more detail may be found in Ref. [33], where the derivation of other coefficients ζ_k, $k=3,\dots,7$ is also given.

Substituting into Eq. (134) the expression for $f_\zeta^{(2)}$, Eq. (135), and for $f^{(1)}$, Eq. (63), and collecting terms corresponding to the same field gradients, we obtain a system of equations for the functions $B_k(n,T,V)$ $k=1,\dots,7$. The coefficients B_1 and B_2 correspond to the linear part of $f_\zeta^{(2)}$ with respect to the fields gradients. Therefore, only the term $\vec{V}\cdot\vec{\nabla}f^{(1)}$ on the right-hand side of Eq. (134) contributes to the equations for B_1 and B_2 [10]. Moreover, due to symmetry requirements only the part

$$-\frac{2m}{15T^3 n}(\kappa\nabla^2 T+\mu\nabla^2 n)\vec{S}\cdot\vec{V}f_M \tag{140}$$

of the term $\vec{V}\cdot\vec{\nabla}f^{(1)}$ contributes to the cooling coefficient $\zeta^{(2)}$ and should be taken into account (see Refs. [10,33] for detail). The equations for B_1 and B_2 then decouple from those for the other coefficients B_k:

$$\zeta^{(0)}T\frac{\partial B_1}{\partial T}+\left(\zeta^{(0)}+T\frac{\partial\zeta^{(0)}}{\partial T}\right)B_1+I(f^{(0)},B_1) +I(B_1,f^{(0)})+\zeta_1 T\frac{\partial f^{(0)}}{\partial T} =\kappa\left(\frac{2}{3nT}T\frac{\partial f^{(0)}}{\partial T}-\frac{2m}{15T^3 n}f_M\vec{S}\cdot\vec{V}\right) \tag{141}$$

and

$$\zeta^{(0)}T\frac{\partial B_2}{\partial T}+I(f^{(0)},B_2)+I(B_2,f^{(0)})+\zeta_2 T\frac{\partial f^{(0)}}{\partial T} \times+\frac{T}{n}\zeta^{(0)}B_1=\mu\left(\frac{2}{3nT}T\frac{\partial f^{(0)}}{\partial T}-\frac{2m}{15T^3 n}f_M\vec{S}\cdot\vec{V}\right). \tag{142}$$

Equations (141) and (142) are similar to the corresponding equations in Ref. [10] for $\varepsilon=\text{const}$, where $T\partial\zeta^{(0)}/\partial T$

$=\zeta^{(0)}/2$. We substitute Eq. (136) into the above equations, multiply them by V^4 and integrate over $\vec{V}$. As a result, we arrive at equations for the coefficients b_1 and b_2,

$$b_1\left(\frac{3\zeta^{(0)}}{B}+\frac{T}{B}\frac{\partial\zeta^{(0)}}{\partial T}+\frac{2}{3}\Omega_\zeta^{(1)}H(a_2)+\frac{4}{15}\Omega_\zeta^{(2)}\right)+\frac{\zeta^{(0)}T}{B}\frac{\partial b_1}{\partial T}=\frac{4\kappa}{3BnT}[H(a_2)-7],$$

$$b_2\left(\frac{2\zeta^{(0)}}{B}+\frac{2}{3}\Omega_\zeta^{(1)}H(a_2)+\frac{4}{15}\Omega_\zeta^{(2)}\right)+\frac{\zeta^{(0)}T}{B}\frac{\partial b_2}{\partial T}+\frac{T\zeta^{(0)}}{Bn}b_1=\frac{4\mu}{3BnT}[H(a_2)-7], \tag{143}$$

where we use the notations

$$H(a_2)\equiv 1+a_2+\frac{1}{2}T\frac{\partial a_2}{\partial T}, \tag{144}$$

and introduce the coefficient

$$\Omega_\zeta^{(2)}\equiv\int d\vec{c}_1\int d\vec{c}_2\int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|\tilde{f}^{(0)}(c_1)\times\phi(c_2)S_2(c_2^2)\Delta(c_1^4+c_2^4). \tag{145}$$

The calculation of this coefficient (see the Appendix) yields

$$\Omega_\zeta^{(2)}=-4\sqrt{2\pi}\left(1+\frac{1}{16}a_2\right)-\left(\frac{903}{125}-\frac{567}{6250}a_2\right)\omega_0\delta'+\left(\frac{476\,973}{44\,000}-\frac{4\,459\,833}{35\,200\,000}a_2\right)\omega_1\delta'^2. \tag{146}$$

Equations (143) are similar to the analogous equations in Ref. [10].

We seek the solution for b_1, b_2 as expansions in terms of δ':

$$b_1=\frac{\kappa_0}{BnT}(\delta' b_{1,1}+\delta'^2 b_{1,2}+\cdots),$$

$$b_2=\frac{\kappa_0}{Bn^2}(\delta' b_{2,1}+\delta'^2 b_{2,2}+\cdots). \tag{147}$$

Substituting Eqs. (147) into Eqs. (143) together with expressions (100) and (123) for η and κ, Eq. (28) for a_2 and Eqs. (139) and (146) for $\Omega_\zeta^{(1/2)}$ and equating terms of the same order in δ', we find the coefficients $b_{1,1/2}$ and $b_{2,1/2}$. These coefficients are then used to find the cooling coefficients ζ_1 and ζ_2, according to Eq. (137). The result reads

$$\zeta_1=\frac{1747}{80\,000}\frac{\omega_0^2}{\pi}\delta'^2\frac{\kappa_0}{nT}, \tag{148}$$

$$\zeta_2=\frac{19}{1000}\frac{\omega_0^2}{\pi}\delta'^2\frac{\kappa_0}{n^2}. \tag{149}$$

As we see from the last equations these coefficients are of the second order with respect to the small parameter δ'.

Very similar considerations may be performed to derive the other coefficients b_k, $k=3,\dots,7$ and correspondingly the cooling coefficients ζ_k. The computation is straightforward but rather lengthy and we wish to give here only the final results, referring for detail to Ref. [33]:

$$\zeta_3=\left[\frac{96}{3125}\omega_0\delta'+\left(\frac{108\,759}{6\,250\,000}\sqrt{\frac{2}{\pi}}\omega_0^2-\frac{357}{12\,500}\omega_1\right)\delta'^2\right]\frac{Bm\kappa_0^2}{3n^2T^3}, \tag{150}$$

$$\zeta_5=\frac{228}{15\,625}\sqrt{\frac{2}{\pi}}\omega_0^2\delta'^2\left(\frac{Bm\kappa_0^2}{3n^3T^2}\right), \tag{151}$$

$$\zeta_6=\left[\frac{358}{225}\omega_0\delta'+\left(\frac{979\,549}{5\,400\,000}\sqrt{\frac{2}{\pi}}\omega_0^2-\frac{253}{150}\omega_1\right)\times\delta'^2\right]\frac{B\eta_0}{3n^2T^2}, \tag{152}$$

$$\zeta_7=-\left[\frac{5152}{675}\omega_0\delta'+\left(\frac{771\,053}{900\,000}\sqrt{\frac{2}{\pi}}\omega_0^2-\frac{1786}{225}\omega_1\right)\times\delta'^2\right]\frac{B\eta_0}{3n^2T^2}. \tag{153}$$

The second-order coefficient ζ_4 is not given, since it is of the order of $\zeta_4\sim O(\delta'^3)$. It is interesting to note that while the coefficients ζ_1, ζ_2, ζ_4, and ζ_5 do not have terms linear in δ', the other coefficients ζ_k do have the linear contribution. We wish also to emphasize that for practical applications the second-order cooling coefficient $\zeta^{(2)}$ is unimportant since it depends on the product of two small values, on the dissipative parameter δ', and on second-order gradients.

VII. TWO-DIMENSIONAL GRANULAR GAS

So far we have restricted ourselves to the three-dimensional systems although the calculations are identical for general dimension d. Of particular interest is the case $d=2$. Since molecular dynamics simulations are frequently performed for the two-dimensional systems, in this section we present the results for the two-dimensional gases of viscoelastic particles. We wish to stress that these systems are, in fact, quasi-two-dimensional, since we still use the coefficient of restitution Eq. (1) for colliding spheres. Hence, we assume that the motion of the spherical particles of the gas is restricted to a two-dimensional surface.

The hydrodynamic equations for the two-dimensional gases have the form

$$\frac{\partial n}{\partial t}+\vec{\nabla}\cdot(n\vec{u})=0,$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} + (nm)^{-1}\vec{\nabla}\cdot\hat{P} = 0,$$

$$\frac{\partial T}{\partial t} + \vec{u}\cdot\vec{\nabla}T + \frac{1}{n}(P_{ij}\nabla_j u_i + \vec{\nabla}\cdot\vec{q}) + \zeta T = 0, \qquad (154)$$

with the pressure tensor and the heat flux

$$P_{ij} = nT\delta_{ij} - \eta(\nabla_i u_j + \nabla_j u_i - \delta_{ij}\vec{\nabla}\cdot\vec{u}),$$

$$\vec{q} = -\kappa\vec{\nabla}T - \mu\vec{\nabla}n. \qquad (155)$$

Correspondingly, the transport coefficients read

$$\eta = \eta_0(1 + \delta'\tilde{\eta}_1 + \delta'^2\tilde{\eta}_2 + \cdots), \qquad (156)$$

where $\delta'(t) = \delta[2T(t)/T_0]^{1/10}$ and

$$\eta_0 = \frac{1}{2\sigma}\sqrt{\frac{mT}{\pi}},$$

$$\tilde{\eta}_1 = \frac{29}{640}\frac{\sqrt{2\pi}}{\pi}\omega_0 \approx 0.234,$$

$$\tilde{\eta}_2 = \frac{111}{160\,000}\frac{\omega_0^2}{\pi} + \frac{569}{14\,080}\frac{\omega_1\sqrt{2\pi}}{\pi} \approx 0.308. \qquad (157)$$

Similarly,

$$\kappa = \kappa_0(1 + \delta'\tilde{\kappa}_1 + \delta'^2\tilde{\kappa}_2 + \cdots), \qquad (158)$$

with

$$\kappa_0 = \frac{2}{\sigma}\sqrt{\frac{T}{\pi m}},$$

$$\tilde{\kappa}_1 = -\frac{433}{3200}\frac{\sqrt{2\pi}\omega_0}{\pi} \approx -0.700,$$

$$\tilde{\kappa}_2 = \frac{1}{\pi}\left(\frac{1\,749\,573}{12\,800\,000}\omega_0^2 + \frac{95\,619}{70\,400}\sqrt{2\pi}\omega_1\right) \approx 11.89 \qquad (159)$$

and

$$\mu = \frac{\kappa_0 T}{n}(\delta'\tilde{\mu}_1 + \delta'^2\tilde{\mu}_2 + \cdots),$$

$$\tilde{\mu}_1 = \frac{7}{20}\frac{\omega_0\sqrt{2\pi}}{\pi} \approx 1.811,$$

$$\tilde{\mu}_2 = \frac{1}{\pi}\left(-\frac{7411}{80\,000}\omega_0^2 + \frac{7}{40}\sqrt{2\pi}\omega_1\right) \approx 0.056. \qquad (160)$$

The zero-order cooling coefficient $\zeta^{(0)}$ reads for a two-dimensional gas of viscoelastic particles

$$\zeta^{(0)} = \sqrt{\frac{2T}{m}}n\sigma\left(\frac{1}{2}\omega_0\delta' - \tilde{\omega}_2\delta'^2 + \cdots\right),$$

$$\tilde{\omega}_2 = \frac{1}{2}\omega_1 + \frac{9\sqrt{2\pi}}{500\pi}\omega_0^2 \approx 5.246, \qquad (161)$$

with the numerical constants $\omega_{0/1}$ given by Eq. (20). The corresponding second-order cooling coefficients are

$$\zeta_1 = \frac{69}{1600}\frac{\omega_0^2}{\pi}\delta'^2\frac{\kappa_0}{nT}, \qquad (162)$$

$$\zeta_2 = \frac{21}{250}\frac{\omega_0^2}{\pi}\delta'^2\frac{\kappa_0}{n^2}, \qquad (163)$$

$$\zeta_3 = \left[\frac{3}{125}\omega_0\delta' + \left(\frac{75\,759}{4\,000\,000}\sqrt{\frac{2}{\pi}}\omega_0^2 - \frac{357}{16\,000}\omega_1\right)\delta'^2\right]\frac{Bm\kappa_0^2}{2n^2T^3}, \qquad (164)$$

$$\zeta_5 = \frac{21}{1250}\sqrt{\frac{2}{\pi}}\omega_0^2\delta'^2\frac{Bm\kappa_0^2}{2n^3T^2}, \qquad (165)$$

$$\zeta_6 = \left[\frac{77}{50}\omega_0\delta' - \left(\frac{33\,071}{400\,000}\sqrt{\frac{2}{\pi}}\omega_0^2 + \frac{659}{400}\omega_1\right)\delta'^2\right]\frac{B\eta_0}{2n^2T^2}, \qquad (166)$$

$$\zeta_7 = -\left[\frac{207}{50}\omega_0\delta' - \left(\frac{149\,061}{400\,000}\sqrt{\frac{2}{\pi}}\omega_0^2 + \frac{1739}{400}\omega_1\right)\delta'^2\right]\frac{B\eta_0}{2n^2T^2}. \qquad (167)$$

As previously, we do not write the second-order coefficient $\zeta_4 \sim O(\delta'^3)$.

VIII. RESULTS AND DISCUSSION

We have derived the hydrodynamics of granular gases of viscoelastic particles. Collisions of viscoelastic particles are characterized by an impact velocity dependent coefficient of restitution. We have used the Chapman-Enskog approach together with an adiabatic approximation for the velocity distribution function, which assumes that the shape of the velocity distribution function follows adiabatically the decaying temperature. We have compared the numerical solutions for temperature $T(t)$ and for the second Sonine coefficient $a_2(t)$ with the corresponding adiabatic approximations and have found good agreement up to intermediate dissipation. To derive the hydrodynamic equations and transport coefficients from the Boltzmann equation we used the standard scheme and account for the additional time dependence of the basic solution. We take into account the time dependence not only due to the thermal velocity, as for $\varepsilon = \text{const}$, but also due to the evolution of the shape of the

TABLE I. Numerical coefficients for Eqs. (168)–(171).

	$3d$	$2d$		$3d$	$2d$
$\tilde{\eta}_1$	0.483	0.234	$\tilde{\eta}_2$	0.094	0.309
$\tilde{\kappa}_1$	0.393	-0.700	$\tilde{\kappa}_2$	4.904	11.893
$\tilde{\mu}_1$	1.229	1.811	$\tilde{\mu}_2$	1.415	0.056
$\tilde{\zeta}_1$	1.078	1.294	$\tilde{\zeta}_2$	-1.644	-2.093

distribution function as given for gases of viscoelastic particles.

Transport coefficients and the cooling coefficient for dilute granular gases of viscoelastic particles read

$$\eta = \eta_0(1+\delta'\tilde{\eta}_1+\delta'^2\tilde{\eta}_2+\cdots), \tag{168}$$

$$\kappa = \kappa_0(1+\delta'\tilde{\kappa}_1+\delta'^2\tilde{\kappa}_2+\cdots), \tag{169}$$

$$\mu = \frac{\kappa_0 T}{n}(\delta'\tilde{\mu}_1+\delta'^2\tilde{\mu}_2+\cdots), \tag{170}$$

$$\zeta^{(0)} = \frac{nT}{\eta_0}(\delta'\tilde{\zeta}_1+\delta'^2\tilde{\zeta}_2+\cdots), \tag{171}$$

where η_0 and κ_0 are the Enskog values for the viscosity and the coefficient of thermal conductivity, and δ' is the time-dependent dissipative parameter. The numerical coefficients $\tilde{\eta}_{1/2}$, $\tilde{\kappa}_{1/2}$, $\tilde{\mu}_{1/2}$, and $\tilde{\zeta}_{1/2}$ are given in Table I. We do not give the corresponding data for the second-order cooling coefficient, since for practical applications it may be neglected.

The dependence on temperature and, therefore, on time of the kinetic coefficients Eqs. (168)–(171) differs significantly from the time dependence of the corresponding coefficients for granular gases of particles which collide with a simplified collisional model $\varepsilon = \mathrm{const}$.

Let us illustrate this with some representative example. We use characteristic parameters of typical granular system, as considered in Ref. [34]: $A = 2\times 10^{-5}$ s, $Y/(1-\nu^2) = 8$ GPa, $R = 0.05$ m, $m = 0.130$ kg, and take the average initial velocity $v_0 = 10^{-1}$ m/s. These parameters yield $\delta = 0.15$, which allows to neglect terms $O(\delta^3)$ and which correspond to the initial restitution coefficient for the average velocity $\varepsilon_{\mathrm{init}} = 0.79$. Using the above theory, we find the values $\eta/\eta_0 = 1.079$, $\kappa/\kappa_0 = 1.187$, $\mu/(\kappa_0 T/n) = 0.232$, and $\zeta^{(1)}/(nT/\eta_0) = 0.130$ at initial temperature $T = T_0$. For the simplified assumption of a constant restitution coefficient these quantities do not depend on time, but only on the value of ε. As the gas evolves, its temperature decreases and if we take the temperature after some time as $T = 10^{-5}T_0$, we obtain for the above quantities: $\eta/\eta_0 = 1.024$, $\kappa/\kappa_0 = 1.032$, $\mu/(\kappa_0 T/n) = 0.065$, and $\zeta^{(0)}/(nT/\eta_0) = 0.050$.

As we see from these estimates, the above reduced quantities are not constant as for the case of $\varepsilon = \mathrm{const}$, but instead change in time. However, while the quantity η/η_0 changes only by about 6% and κ/κ_0 by only 15%, the other two variables change by a factor of 3.5 and a factor of 2.6, respectively. These time dependencies affect the global behavior of force-free granular gases [35].

Under mild preconditions the presented formalism for the derivation of the hydrodynamic equations and the transport coefficients may be applied also to gases of particles whose collision is described by a different impact velocity dependence than given for viscoelastic particles.

APPENDIX: DERIVATION OF THE COEFFICIENTS Ω_η, Ω_κ, AND $\Omega_\zeta^{(1/2)}$

For the evaluation of the numerical coefficient Eq. (96), defined by

$$\Omega_\eta \equiv \int d\vec{c}_1 \int d\vec{c}_2 \int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|\tilde{f}^{(0)}(c_1) \times \phi(c_2)D_{ij}(\vec{c}_2)\Delta[D_{ij}(\vec{c}_1)+D_{ij}(\vec{c}_2)], \tag{A1}$$

we need the factor

$$D_{ij}(\vec{c}_2)\Delta[D_{ij}(\vec{c}_1)+D_{ij}(\vec{c}_2)] = (\vec{c}_1{}'\cdot\vec{c}_2)^2+(\vec{c}_2{}'\cdot\vec{c}_2)^2-(\vec{c}_1\cdot\vec{c}_2)^2 - (\vec{c}_2\cdot\vec{c}_2)^2-\tfrac{1}{3}c_2^2(c_1'^2+c_2'^2-c_1^2-c_2^2). \tag{A2}$$

Similarly, for the coefficient

$$\Omega_\kappa \equiv \int d\vec{c}_1 \int d\vec{c}_2 \int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|\tilde{f}^{(0)}(c_1) \times \phi(c_2)\vec{S}(\vec{c}_2)\cdot\Delta[\vec{S}(\vec{c}_1)+\vec{S}(\vec{c}_2)], \tag{A3}$$

given by Eq. (119), we need

$$\vec{S}(\vec{c}_2)\cdot\Delta[\vec{S}(\vec{c}_1)+\vec{S}(\vec{c}_2)] = \left(c_2^2-\frac{5}{2}\right)[(\vec{c}_1{}'\cdot\vec{c}_2)(c_1')^2+(\vec{c}_2{}'\cdot\vec{c}_2)(c_2')^2 - (\vec{c}_1\cdot\vec{c}_2)c_1^2-(\vec{c}_2\cdot\vec{c}_2)c_2^2]. \tag{A4}$$

With (A2) and (A4), we write for the coefficient Ω_η,

$$\Omega_\eta = \int d\vec{c}_1 \int d\vec{c}_2 \int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|\tilde{f}^{(0)}(c_1)\phi(c_2) \times\Bigg[(\vec{c}_1{}'\cdot\vec{c}_2)^2+(\vec{c}_2{}'\cdot\vec{c}_2)^2-(\vec{c}_1\cdot\vec{c}_2)^2-(\vec{c}_2\cdot\vec{c}_2)^2 - \frac{1}{3}c_2^2(c_1'^2+c_2'^2-c_1^2-c_2^2)\Bigg], \tag{A5}$$

and for the coefficient Ω_κ,

$$\Omega_\kappa = \int d\vec{c}_1 \int d\vec{c}_2 \int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|\tilde{f}^{(0)}(c_1)\phi_2(c_2) \times\left(c_2^2-\frac{5}{2}\right)[(\vec{c}_1{}'\cdot\vec{c}_2)(c_1')^2+(\vec{c}_2{}'\cdot\vec{c}_2)(c_2')^2 -(\vec{c}_1\cdot\vec{c}_2)c_1^2-(\vec{c}_2\cdot\vec{c}_2)c_2^2]. \tag{A6}$$

The coefficients $\Omega_\zeta^{(1)}$ and $\Omega_\zeta^{(2)}$ have similar structure:

$$\Omega_\zeta^{(1)} = \int d\vec{c}_1 \int d\vec{c}_2 \int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|\tilde{f}^{(0)} \times(c_1)\phi(c_2)(\vec{c}_{12}\cdot\vec{e})^2 S_2(c_2^2)(1-\varepsilon^2) \tag{A7}$$

and

$$\Omega_\zeta^{(2)} = \int d\vec{c}_1 \int d\vec{c}_2 \int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|\tilde{f}^{(0)}(c_1) \times\phi(c_2)S_2(c_2^2)(c_1'^4+c_2'^4-c_1^4-c_2^4). \tag{A8}$$

The precollision velocities $\vec{c}_1$, $\vec{c}_2$ as well as after-collision velocities $\vec{c}_1{}'$, $\vec{c}_2{}'$ can be expressed in terms of the center of mass velocity $\vec{C}=(\vec{c}_1+\vec{c}_2)/2$ and the relative velocity $\vec{c}_{12}=\vec{c}_1-\vec{c}_2$ before the collision

$$\vec{c}_1=\vec{C}+\tfrac{1}{2}\vec{c}_{12},$$

$$\vec{c}_2=\vec{C}-\tfrac{1}{2}\vec{c}_{12},$$

$$\vec{c}_1{}'=\vec{C}+\tfrac{1}{2}\vec{c}_{12}-\tfrac{1}{2}(1+\varepsilon)(\vec{c}_{12}\cdot\vec{e})\vec{e},$$

$$\vec{c}_2{}'=\vec{C}-\tfrac{1}{2}\vec{c}_{12}+\tfrac{1}{2}(1+\varepsilon)(\vec{c}_{12}\cdot\vec{e})\vec{e}. \tag{A9}$$

The coefficient of restitution is expressed in terms of the relative velocity $\vec{c}_{12}$ by

$$\varepsilon = 1 - C_1\delta'(t)|\vec{c}_{12}\cdot\vec{e}|^{1/5}+\tfrac{3}{5}C_1^2\delta'^2(t)|\vec{c}_{12}\cdot\vec{e}|^{2/5}+\cdots. \tag{A10}$$

The second Sonine polynomial in the distribution function

$$\tilde{f}^{(0)}(c_1)=\phi(c_1)[1+a_2S_2(c_1^2)], \tag{A11}$$

reads in terms of $\vec{C}$ and $\vec{c}_{12}$,

$$S_2(c_1^2)=\frac{C^4}{2}+\frac{1}{2}(\vec{C}\cdot\vec{c}_{12})^2+\frac{1}{32}c_{12}^4+C^2(\vec{C}\cdot\vec{c}_{12})+\frac{1}{4}C^2c_{12}^2 +\frac{1}{4}c_{12}^2(\vec{C}\cdot\vec{c}_{12})-\frac{5}{2}C^2,-\frac{5}{2}(\vec{C}\cdot\vec{c}_{12})-\frac{5}{8}c_{12}^2+\frac{15}{8}. \tag{A12}$$

If we replace all factors in the integrands of Eqs. (A5)–(A8) by the corresponding expressions in terms of $\vec{C}$ and $\vec{c}_{12}$, we observe that Ω_η, Ω_κ, and $\Omega_\zeta^{(1/2)}$ may be written as a sum of integrals of the structure

$$J_{k,l,m,n,p,\alpha} = \int d\vec{c}_{12} \int d\vec{C} \int d\vec{e}\,\Theta(-\vec{c}_{12}\cdot\vec{e})|\vec{c}_{12}\cdot\vec{e}|^{1+\alpha} \times\phi(c_{12})\phi(C)C^k c_{12}^l(\vec{C}\cdot\vec{c}_{12})^m(\vec{C}\cdot\vec{e})^n(\vec{c}_{12}\cdot\vec{e})^p. \tag{A13}$$

The solution of this integral, in general, dimension d reads [24] for $n=0$

$$J_{k,l,m,0,p,\alpha}=(-1)^p[1+(-1)^m]2^{l+m+p+\alpha+1}\Omega_d^{-1} \times\beta_{p+\alpha+1}\beta_m\gamma_{k+m}\gamma_{l+m+p+\alpha+1}, \tag{A14}$$

for $n=1$

$$J_{k,l,m,1,p,\alpha}=(-1)^{p+1}[1+(-1)^{m+1}]2^{l+m+p+\alpha+1}\Omega_d^{-1} \times\beta_{p+\alpha+2}\beta_{m+1}\gamma_{k+m+1}\gamma_{l+m+p+\alpha+1}, \tag{A15}$$

and for $n=2$

$$J_{k,l,m,2,p,\alpha}=(-1)^p[1+(-1)^m]2^{l+m+p+\alpha+1} \times[(d-1)\Omega_d]^{-1}\gamma_{k+m+2}\gamma_{l+m+p+\alpha+1} \times[(d\beta_{p+\alpha+3}-\beta_{p+\alpha+1})\beta_{m+2} +(\beta_{p+\alpha+1}-\beta_{p+\alpha+3})\beta_m], \tag{A16}$$

where

$$\Omega_d=\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \tag{A17}$$

is the surface of a d-dimensional unit sphere. The coefficients β_m, γ_m read

$$\beta_m=\pi^{(d-1)/2}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+d}{2}\right)},$$

$$\gamma_m=2^{-m/2}\frac{\Gamma\left(\frac{m+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \tag{A18}$$

Following this procedure, we obtain the desired coefficients as given by Eqs. (97), (120), (139), and (146).

The evaluation of the sums which lead to the factors Ω_η, Ω_κ, and $\Omega_\zeta^{(1/2)}$ is straightforward, however, very lengthy. They can be calculated by symbolic algebra [36].

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